MODERN JUNIOR MATHEMATICS
BOOK : : : : TWO

MARIE GUGLE

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BOOK TWO

BY

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PREFACE

Until recently upper elementary and high school work in mathematics was planned for the pupil who was expected to continue it in the university. Although logical, its arrangement was neither psychological nor pedagogical, but some progress has been made recently in adapting the study to the needs and abilities of pupils.

In the junior high or intermediate school, work in mathematics in the seventh, eighth, and ninth grades should be complete in itself and at the same time preparatory to senior high school work. No effort should be made to "finish" arithmetic in the eighth grade and algebra in the ninth, while denying the child the interest and beauty that lie in geometry and trigonometry until his taste for mathematics has been destroyed. Nor will alternate bits of formal algebra, geometry, and trigonometry solve the problem. The result is a mastery of none and a confusion of all.

Experience has proved that the necessary elements of arithmetic can be taught and certain definite skill developed in the first six grades. In the seventh grade business applications of arithmetic with the simplest elements of bookkeeping should be given. In the eighth grade, mensuration should be taught experimentally or through observational geometry, and through that, in a natural and meaningful way, the

very beginnings of algebra. Optional courses should be offered in the ninth grade.

Experience in junior high schools has shown that much of the content and the whole of the organization of subject matter must be changed to make the course of study fit the needs of the pupils.

The definite aims of this study are:

1. To extend the pupil's knowledge of arithmetic through its practical applications in mensuration.

2. To train the hand to use the simple drawing instruments.

3. To familiarize the pupil with common geometric forms.

4. To train him to see geometric forms in nature and in the various buildings and other structures in his surroundings, and to appreciate their use in design.

5. Through experiment and observation to develop the formulas of mensuration.

6. Through a continued study of formulas to introduce general number in a natural way that will give algebraic expressions such a meaning to the pupil that he will use them as convenient and practical tools.

7. To permit a pupil to live so continuously in the atmosphere of geometry that he may be enabled to think naturally and without confusion, in its terms, about its relations.

This book is planned for a year's work in the eighth grade, with the idea that the pupil should advance slowly by doing and thinking for himself. If necessary it can be condensed into one semester's work. It

should always precede and form an introduction to the more formal algebra and geometry.

The nucleus of this book grew out of the author's experience as an instructor in mathematics in a large city high school, in teaching the first book of plane geometry without a text. Four years ago an outline of it was given by the author to the teachers under her supervision because no suitable text in experimental geometry could be found. By way of further suggestion, one or two topics were expanded in more detail and discussed with the teachers. This book is the result of their urgent request for more. Since then, over a score of teachers have used the outline, and their unanimous opinion is that the pupils take an increased delight in mathematics of this kind. The author was in a position to observe the effects of teaching it in various types of schools and found it most gratifying.

The author is very much indebted to Dr. John H. Francis, Superintendent of Schools, Columbus, Ohio, for reading the manuscript and for giving helpful suggestions. She hereby acknowledges her indebtedness also to Miss Meta Philbrick of Mt. Vernon Intermediate School and to Miss Amy Preston of Roosevelt Intermediate School, Columbus, for their coöperation and assistance in gathering problems and program material.

Marie Gugle

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BOOK TWO

EQUIPMENT FOR MATHEMATICS DEPARTMENTS

IN INTERMEDIATE AND HIGH SCHOOLS

1.	Graph chart	1 per room
2.	Yard or meter sticks	2 dozen per room
3.	Balls of cord	2 balls per room
4.	Parallel rulers	1 per room
5.	Protractor for blackboard	1 per room
6.	100-ft. tape line	1 per room
7.	Set of mensuration blocks	1 per room
8.	Scissors	2 dozen per room
9.	Equation balance	
10.	2 large wooden triangles . . . . (90–60–30 and 90–45–45 degrees)	1 set per room
11.	Slated globe (8″ or 12″ mounted and detachable)	1 per room
12.	Adjustable geometric models . .	1 set per teacher of solid geometry
13.	Inexpensive transit	1 per building
14.	Supply of cardboard, paste, and heavy paper	
15.	Books	1 copy per building

(a) History of Mathemathics — Ball
(b) Scrapbook of Mathematics — White
(c) Mathematical Wrinkles —Jones
(d) Mathematical Recreations — Ball
(e) Flatland — Abbott
(f) Magic Squares and Cubes — Andrews
(g) The Hindu-Arabic Numerals — Smith & Karpinski
(h) Number Stories of Long Ago — Smith

EQUIPMENT FOR EACH PUPIL

1. Combination ruler and protractor
   (of transparent celluloid marked in inches and centimeters; some inches divided into sixteenths and some into tenths)
2. Compass that uses a pencil
3. Graph paper
4. Two right triangles — (90–60–30 and 90–45–45 degrees)

MODERN JUNIOR MATHEMATICS

BOOK TWO

CHAPTER ONE

FORM AND MEASUREMENT

A. FORM

HERETOFORE your interest in mathematics has centered around computation, especially in making with accuracy and speed the calculations used in everyday business. In order to retain the skill you have developed, it will be necessary for you to practice.

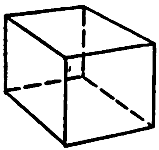
But there is a great deal more in mathematics than mere calculation with figures. This is a very important and necessary part of all mathematics, although it is the newest part, for the figures which make our numbers were not known in Europe until the thirteenth century. Nevertheless fifteen hundred and more years before, there lived some of the most famous mathematicians of all times. A knowledge of the mathematics developed by these men enables us to survey our land, to build our houses, buildings, bridges, and ships, to design our furniture and our art, and to make our patterns for various kinds of manufactures.

We live in a world of people no two of whom are exactly alike, although all faces have the same features

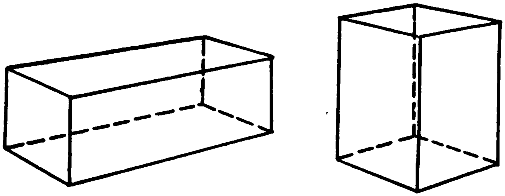
of forehead, nose, eyes, ears, mouth, and chin. We also live in a world of things which, though they are many and varied, have a few common features or forms.

I. Blocks

The cube and oblong blocks are two of these forms which every one knows. But, although they are so



CUBE



OBLONG BLOCKS

familiar, most people cannot answer offhand a few simple questions about them.

Without examining these blocks further, try to answer the following questions without hesitation:

1. Name different objects around you that have these shapes.

2. How many dimensions has each block?

3. What are these dimensions?

4. How many faces has each block?

5. How many edges has each?

6. How many corners has each?

7. (a) How does the position of any one face compare with the one opposite it?

(b) Two such opposite faces or surfaces that are everywhere the same distance apart are *parallel*.

(c) The word *parallel* is derived from two Greek words which mean *beside one another*. A mathema-

tician writes *parallel* thus, $||$. This sign, $||$, may mean *parallel* or *is parallel to*. A small *s* beside it makes it plural. $||$s means *parallels.*

8. What is the shape of the faces of a cube? of an oblong?
9. How many edges of a cube are equal?
10. How many edges of an oblong are equal?
11. Compare the lengths of the opposite edges of any one face; the position.

B. MEASUREMENT

I. Perimeter

1. (a) The word *perimeter* comes from two Greek words which mean the *measure around.* The perimeter of any object is the measure around it.
   (b) You are familiar with the word *meter* in the terms *gas meter*, *speedometer* (speed measure), and *thermometer* (heat measure).
2. (a) Find the perimeter of a face of the cube.
   (b) How many edges must you measure to find the perimeter of one face?
   (c) Is the following statement true? If so, why? The *perimeter* of a face of a cube or of a square is *equal* to *four times* the length of one *edge.*
3. (a) Is this true for any face of the oblong? Why?
   (b) How many edges of the oblong must you measure to know the perimeter of one of its faces?
4. (a) To distinguish certain faces of a cube or of an oblong from others, definite names are given.
   (b) The face on which the object stands is called the *base* or the lower base. The opposite one, or the top, is the upper base.
   (c) The faces around the sides are called *lateral* faces. *Lateral* comes from a Latin word which means *side.*

5. (a) Can you find the perimeter of the base of an oblong from the length and width only? How?

(b) Show that the *perimeter* of the base of an oblong or of a rectangle is *equal* to *two times* the *sum* of its *length* and *width.*

6. (a) Find the perimeter of a cube whose edge is 3 inches; 7 inches; $5\frac{1}{2}$ inches.

(b) Find the perimeter of the base of an oblong that is 6 inches long and 4 inches wide; of one that is 8 feet long and $3\frac{1}{2}$ feet wide; of one that is $5\frac{1}{2}$ inches by $2\frac{1}{2}$ inches.

7. In the statements, sections 12 (c) and 15 (b), let the first letters of the main words be used in place of the words, thus:

(a) For square, $P = 4 \times e$

(b) For rectangle, $P = 2 \times (l + w)$

(c) By putting parentheses ( ) around $l + w$, we show that we must find the sum before multiplying by 2.

8. (a) Such a short way of writing a statement is a *formula.*

(b) Carpenters, surveyors, engineers, and many other craftsmen find it very convenient to use formulas for several reasons:

1. Because formulas take much less writing and are more quickly read than the English statement in full.

2. Because such formulas are true for all rectangles and squares.

9. (a) If the edge of a cube or $e = 3$ in., then by putting 3 in the place of e in the formula, we have for the cube or square,

$$\begin{aligned} P &= 4 \times e \\ &= 4 \times 3 \\ &= 12 \text{ inches} \end{aligned}$$

(b) If a rectangle is four inches long and three inches wide, then we have

for the rectangle,
$$\begin{aligned} P &= 2 \times (l + w) \\ &= 2 \times (4 + 3) \\ &= 2 \times 7 \\ &= 14 \text{ inches} \end{aligned}$$

(c) This process is called *substituting numerical values* for the letters in a formula.

(d) Instead of writing out the words "for a square" or "for a rectangle" to show which particular perimeter we mean, we may draw a little square or a little rectangle directly after the letter P.

$P_{\square}$ means *perimeter of a square.*

$P_{\text{▭}}$ means *perimeter of a rectangle.*

10. (a) We have seen that

$$\begin{aligned} P_{\square} &= e + e + e + e \\ &= 4 \times e \end{aligned}$$

Therefore, $4 \times e$ means that e has been added to itself four times, or e has been multiplied by 4. It is shown in

$$\begin{aligned} 12 &= 3 + 3 + 3 + 3 \\ &= 4 \times 3 \end{aligned}$$

(b) A number used (as the 4) to show how many times another number (as the e or 3) has been *added* to *itself* is a *coefficient.*

Since multiplication is only shortened addition, the coefficient shows that the other part of the expression is to be multiplied by the coefficient.

(c) When we multiply two arithmetical numbers together we have to use the *times* sign ($\times$), as 4×3; but when one number is a letter, as $4 \times e$, we may omit the times sign and write it $4\,e$ (read "four e").

(d) If the sign were omitted between the 4 and 3, what would the number mean?

(e) Instead of using the cross ($\times$) to indicate multiplication, we may use a dot placed between the numbers and above the line of writing. If it were placed on the line of writing, with what might it be confused?

$3 \cdot 7$ means 3 times 7, just as 3×7.

The use of the dot avoids the confusion of the cross with the letter x.

What is the value of $2 \cdot 2 \cdot 3 \cdot 7$? of $3 \cdot 4 \cdot 7 \cdot 2 \cdot 5$?

11. What are the coefficients of the following?

5 e, 6 cwt., 10 T., $2\frac{1}{2}$ doz., 3 lb., 2 ft., 25¢.

12. (a) What do C and M stand for in Roman numerals?

(b) How many sheets of letterheads are there in 2 M? In 4 C?

(c) How many are there in 2 M + 4 C?

13. How many pounds of hay are in $7\frac{1}{2}$ T.?

14. In two mows there are $2\frac{1}{2}$ T. and $3\frac{3}{4}$ T. of hay, respectively. How many T. are in both? how many pounds?

15. By using the formula for a square, find the perimeter of a cube whose edge is (a) 4 in.; (b) $7\frac{1}{2}$ in.; (c) $\frac{3}{4}$ ft.

16. By using the formula for a rectangle, find the perimeter of the base of an oblong whose length and width, respectively, are (a) 7 in. and 5 in.; (b) $6\frac{1}{2}$ in. and $4\frac{1}{4}$ in.

17. (a) Estimate the length and width of your desk and find the perimeter.

(b) Measure with your ruler and find the perimeter more exactly.

18. Estimate and then measure the perimeter of (a) your book; (b) your school room; (c) your teacher's desk; (d) a picture on the wall; (e) a door; (f) a window; (g) any other square or rectangular objects that may be measured conveniently.

CHAPTER TWO

SURFACES OF COMMON SOLIDS

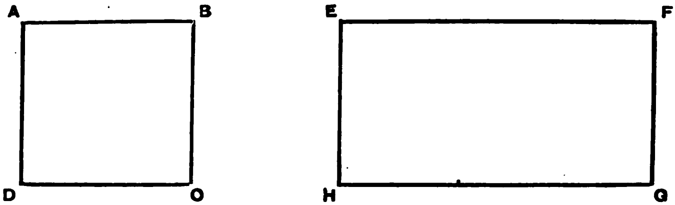
A. STUDY OF CUBES AND OBLONGS

I. Squares and Rectangles

1. Paper ruled into small squares is called squared paper. Each pupil should have a supply of it.

2. On squared paper draw a picture of one face of a cube; of an oblong.

3. We have observed that the opposite edges are parallel. For convenience, let us name the four corners with



different letters, so that we may tell which corner we are talking about.

4. (a) Let us call the side between the corners at *A* and *B*, the line *AB*.

(b) Read the names of the other sides.

5. (a) Two sides that meet at the same corner or point are called *adjacent* sides; as, *AB* and *AD*.

(b) The word *adjacent* means *lying next to.*

(c) Name the pairs of adjacent sides in each figure.

(d) Name the pairs of opposite sides in each.

II. Angles

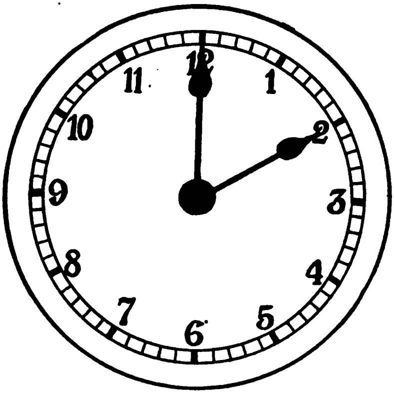
1. (a) Whenever two lines are drawn from the same point an opening is formed called an *angle.*

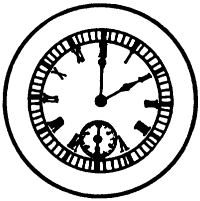
(b) *Angle* means *corner*.

(c) An angle is the amount of opening between two straight lines that meet.

(d) The size of the angle increases as the lines separate from each other.

The hands of a clock or watch always form an angle which is constantly changing in size.





The hands on the face of a large clock make the same angle at 2 o'clock as the hands of a small watch. The length of the hands has nothing to do with the size of the angle. The size depends solely on how far apart the hands are.

2. (a) At 3 o'clock and 9 o'clock the angles which the hands form are *right angles*.

(b) At 6 o'clock the hands form a straight line and the angle becomes a *straight* angle.

3. (a) Fold a sheet of paper smoothly.

(b) Fold it again, putting the two edges together carefully.

(c) The four angles or quarters are now equal, for they exactly fit each other.

(d) Unfold the paper. The two creases or lines cross each other, so that the four angles are equal and are *right angles*.

(e) If one line meets another line so that the two angles are equal, the angles are called *right angles*.

(f) Two lines that meet so as to make equal angles or right angles are *perpendicular* to each other.

4. (a) Use this folded paper to measure the angles of the square and rectangle.

(b) What kind of angles are they?

(c) What is the sum of all the angles in each figure?

(d) The word *rectangle* means *right angle*. Therefore, if a four-sided figure has all its angles right angles, it is a *rectangle*.

(e) The sign for right angle is *rt.* $\angle$; in the plural it is *rt.* $\measuredangle$s.

(f) The sign for perpendicular is $\perp$, or $\perp$s for the plural. Sometimes $\perp$ means *is perpendicular to*, as $AB \perp CD$ means that the line AB is perpendicular to the line CD.

5. (a) Just as a foot ruler is divided into twelve smaller parts called inches, so a right angle is divided into ninety smaller angles called degrees.

(b) A right angle is said to contain ninety degrees (90°).

6. (a) Unfold your angle paper once.

(b) How does this angle compare with that made by the hands of a clock at 6 o'clock?

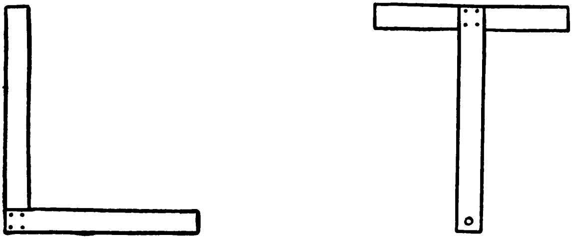
(c) What is the name of this angle?

(d) How many right angles are in it?

(e) How many degrees are in a straight angle?

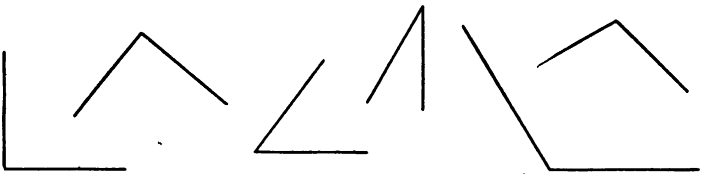
7. To draw right angles and perpendicular lines, car-

penters use a steel instrument called a square or T square, depending on whether its shape is that of the letter L or T.



CARPENTER'S SQUARE T SQUARE

8. (a) At a quarter of nine o'clock the hands are almost together. At 9 o'clock they form a right angle. In that fifteen minutes many different sized angles were formed, but all were *less than* a right angle.
   (b) Any angle whose size is *less than* a right angle is an *acute angle.*
   (c) *Acute* means *sharp.*
   (d) Draw an acute angle and see why it is properly named a "sharp" angle.



RIGHT ANGLES ACUTE ANGLES OBTUSE ANGLES

9. (a) What is the size of the angle made by the hands of a clock at 9 o'clock?
   (b) At 9.15 it is almost but not quite a straight angle.
   (c) In that fifteen minutes many different sized angles were formed, but all were *greater than* a right angle and *less than* a straight angle.

(d) Any angle that is *greater than* a right angle and *less than* a straight angle is an *obtuse angle.*

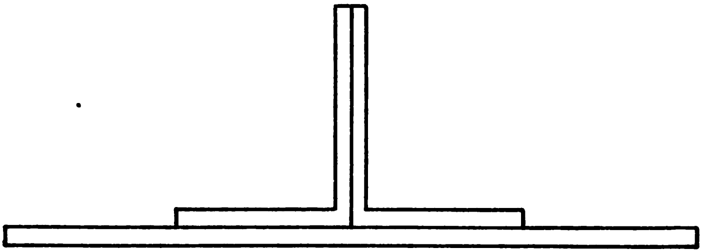
(e) *Obtuse* means *blunt.*

(f) Draw an obtuse angle and see why it is properly named a "blunt" angle.

10. (a) A carpenter tests his squares, that is, he sees whether they are true right angles or not, by placing two of them together on a straight edge, as shown in the figure.

(b) If the two edges exactly fit when thus placed, the outside angles are true right angles. Why?

(c) How may the inside angle of a square be tested?



11. What kind of angle do the hands of a clock make at 3 o'clock?

12. Do they make the same kind at a quarter past 12?

13. At what times do they make right angles?

14. Do they make a straight angle at 12.30?

15. What kind of angle is made by the hands of a clock at 3.30, 3.35, 1.00, 1.30, 4.30, 8.00, 8.55, 8.40 o'clock?

B. OTHER QUADRILATERALS

I. Parallelograms

1. (a) How many sides has a square?

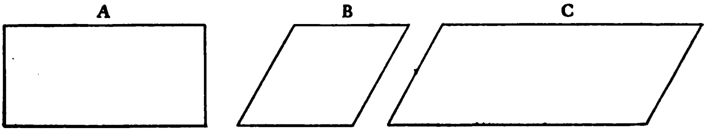
(b) Are all of the sides of a square equal?

(c) Are the opposite sides parallel?

(d) Are all of the angles right angles?
(e) Are *all* of these features necessary for a square?
(f) Name all the necessary features of a square.

2. (a) Try making four-sided figures with only two of the other features, as given in (*b*), (*c*), and (*d*).
(b) Make a four-sided figure which has only the feature (*c*).

3. (a) Which features are found and which are lacking in the following figures?



RECTANGLE RHOMBUS RHOMBOID

(b) All of these figures are parallelograms because each has two pairs of parallel sides.
(c) *Parallelogram* means *parallel drawing.*
(d) The sign for parallelogram is ▱. For the plural it is $\boxed{S}$.
(e) Fill in these blanks to make a correct definition:

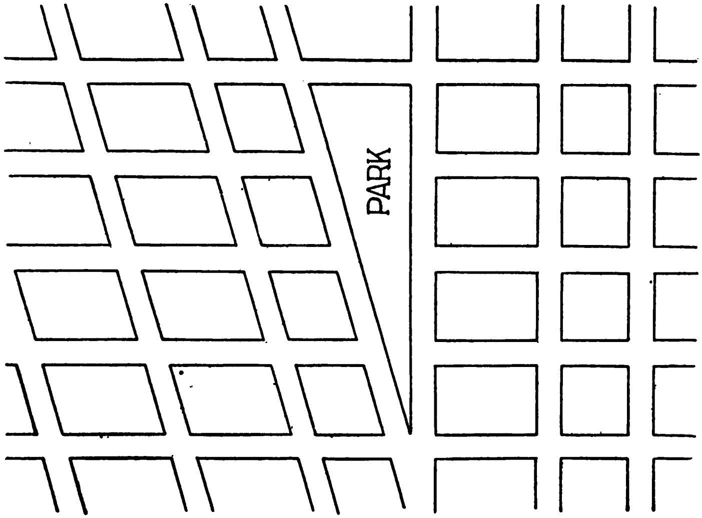
A parallelogram is a figure inclosed by — pairs of — lines.

(f) Is a square a parallelogram?
(g) Is every square a rectangle?
(h) Is every rectangle a square?
(i) We always give the special name of *square, rectangle,* or *rhombus* to each of the particular kinds of parallelograms, but seldom use the name *rhomboid.* When we speak of a *parallelogram* we mean the general form or rhomboid.

4. (a) How many objects can you find having these shapes?

(b) What is the shape of your schoolroom? of your school yard? of the pages of your book? of the door?

5. Below is a map of a section of a city. What are the shapes of the various city blocks?



C. MEASUREMENT OF AREA

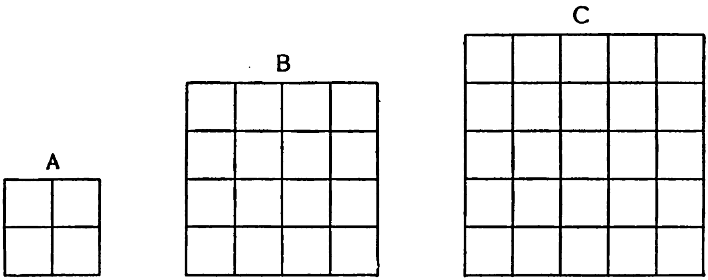
I. Area of Squares and Rectangles

1. On squared paper draw a picture of one face of a cube whose edge is one inch. This picture is called a *square inch* and is used to measure surface or area.

(If metric units are desired, do the same with one centimeter.)

2. Draw a square whose edge is 2 inches. Count the number of square inches in it. This number is called the measure of the area or simply the *area* of the square.

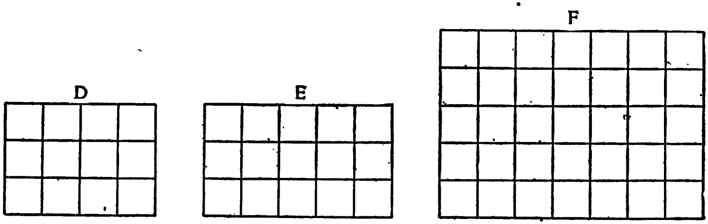
3. Draw squares with different edges and count the square units in area.



Area of square A = $2 \cdot 2$ or 2^2 or $4 \cdot 1$ sq. in. = 4 sq. in.
" " square B = $4 \cdot 4$ or 4^2 or $16 \cdot 1$ sq. in. = 16 sq. in.
" " square C = $5 \cdot 5$ or 5^2 or $25 \cdot 1$ sq. in. = 25 sq. in.

4. By measuring one edge of a square, how may you find the number of square units in its area?

5. On squared paper, draw several rectangles and count the square units in their areas.



Area of rectangle D = $4 \cdot 3$ or $12 \cdot 1$ sq. in. = 12 sq. in.
" " " E = $5 \cdot 3$ or $15 \cdot 1$ sq. in. = 15 sq. in.
" " " F = $7 \cdot 5$ or $35 \cdot 1$ sq. in. = 35 sq. in.

6. (a) Thus we find that the *area* or surface of a rectangle or of a square *equals* the *length times* the *width.*

(b) In this statement let the first letters of the main words be used in their places. Then we have

$$S_{\square} = l \cdot w$$

7. (a) What do we call such a statement?
(b) Does this statement hold true for all rectangles?

8. (a) Measure your desk, your book, and your school room and find their areas.
(b) What units will you use for measuring each of these?

9. (a) In what kind of rectangle is the length equal to the width?
(b) If e stands for the edge of a cube, we may say,
area of one surface = edge × edge,

or

$$S_{\square} = e \cdot e = e^2$$

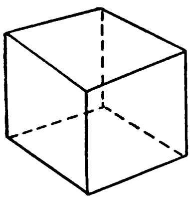
10. If the edge of a cube is 3 inches, then

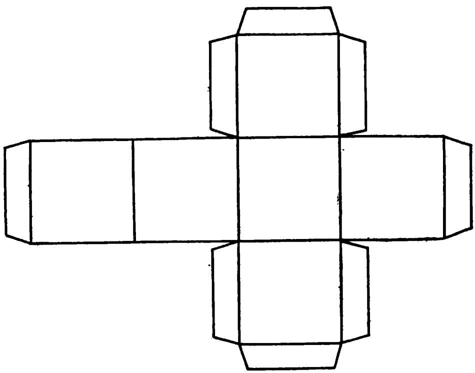
$$S_{\square} = 3 \cdot 3 = 3^2 = 9 \textit{ sq. in.}$$

11. Find the area of a face if the edge of cube is 6 in.; $4\frac{1}{2}$ in.

II. Surface of Cube and Oblong

1. Imagine the surface of the cube as being a very thin covering that may be peeled off as an onion skin. Run





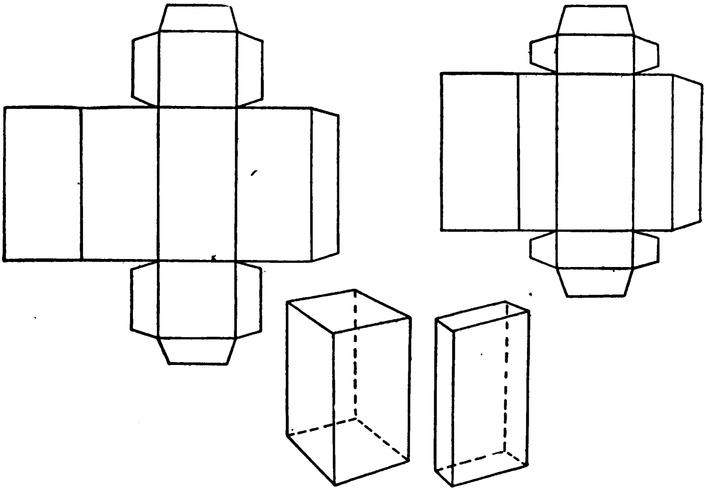
a knife down one edge and peel off that face without detaching it. Do the same for the opposite face. Then peel off the rest in one piece. Make a pattern of this surface on

stiff paper. Put on necessary flaps, cut out, paste, and put together.

2. Do the same for the oblong.

3. Another name for this oblong is *parallelopiped.* A cube is also a parallelopiped, but all of its edges are equal.

4. Make two patterns, one for an oblong with square bases and one with rectangular bases.



A B

5. (a) How many faces has a cube?

(b) How do the different faces compare in size and shape?

(c) How would you find the total surface, that is, the sum of the areas of all of the faces?

(d) The total surface of cube $= 6 \times$ area of one face.

$$\text{For cube, Tot. } S = 6\,e^2$$

If the edge is 3 in., then

$$\begin{aligned}\text{Tot. } S &= 6 \cdot 3^2\\ &= 54 \text{ sq. in.}\end{aligned}$$

6. How many faces of a cube are lateral faces? Can you see the reason for the following formula?

$$\text{For cube, Lat. } S = 4\,e^2$$

7. (a) How many different dimensions has the first oblong, *A?* the second, *B?* (b) Name them.

(c) What is the shape of the pattern of the lateral surface?

(d) What is the total length of this rectangle?

(e) What dimension of the oblong is the width of this rectangle?

(f) The formula for the lateral surface is

$$\begin{aligned} \text{Lat. } S &= 2\cdot(l + w)\cdot h \\ \text{or,} \quad &= 2h(l + w) \end{aligned}$$

(g) Put this formula into a complete English sentence and see if it is true for both figures.

(h) Can you think of a practical problem in which you would want to find the lateral area only? Would the painting of a fence be such a problem?

(i) How many bases has an oblong?

(j) How can you find the area of each?

(k) How can you find the total area of the parallelopiped?

(l) Lat. $S = 2\cdot(l+w)\cdot h$ or Lat. $S = 2\,h(l+w)$

2 Bases, $S = 2\cdot l\cdot w$	Bases, $S = 2\,lw$
Tot. S = sum of all parts	Tot. S = sum of all parts

8. Find the total area of a chest if its length is 10 ft., its width 4 ft., and its height 3 ft.

Solution.

$$\begin{aligned} \text{Lat. } S &= 2\,h(l + w) = 2\cdot 3(10 + 4) \\ &= 2\cdot 3\cdot 14 = 84 \text{ sq. ft.} \\ \text{2 Bases, } S &= 2\,lw = 2\cdot 10\cdot 4 = 80 \text{ sq. ft.} \\ \hline \text{Total } S &= 164 \text{ sq. ft.} \end{aligned}$$

9. Another formula for total surface is,

$$\begin{aligned}\text{Tot. } S &= 2(lw + lh + wh)\\ &= 2(10\cdot 4 + 10\cdot 3 + 4\cdot 3)\\ &= 2(40 + 30 + 12)\\ &= 2(82)\\ &= 164 \text{ sq. ft.}\end{aligned}$$

Show why this formula is correct.

10. The second method is more convenient when only the total area is desired. If the area of the lateral surface and one base only is required, the first method is the better.

11. A flower box is 8 ft. long, 10 in. wide, and 8 in. high, inside measurement. How many square feet of tin will be required to line the box? No allowance is made for waste.

12. How many square yards of cement pavement are needed for a walk $4\frac{1}{2}$ ft. wide in front of a lot 50 ft. wide?

13. (a) An athletic field is inclosed by a board fence 10 ft. high. The field is 600 ft. $\times$ 400 ft. One gallon of paint covers 250 sq. ft. with two coats. How many gallons will be needed to paint the fence?

(b) A painter calls 100 square feet a square. If he can paint a square of fence in one hour, what will be the cost of labor in painting the fence? Inquire of a painter the scale of wages.

(c) What will be the total cost?

14. (a) How many square feet are in the side walls of your school room?

(b) Find their area with the doors and windows taken out.

15. (a) A house is 40 ft. $\times$ 32 ft. $\times$ 18 ft. How many squares in its surface? (Use the nearest integer.)

(b) Allow 14 squares for the dormer windows, cornice, and porch. How many gallons of paint are

needed? (No allowance is made for doors and windows because they take more paint than flat surfaces.)

(c) Find the cost of the paint at $4.25 per gal.

(d) It takes a painter two hours to paint a square of a house. Find the cost of labor at $.70 per hour. Find the total cost.

(e) How many days are needed for the work?

(f) What does the painter earn in an 8-hour day? in a week?

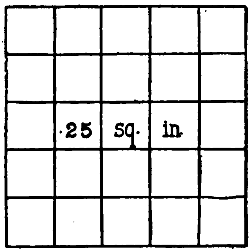
D. SQUARE ROOT

I. Study of Numbers and Factors

1. (a) On squared paper draw a square inclosing 9 sq. in.

(b) Draw one inclosing 25 sq. in.

(c) How long is the edge of each?



2. Since the area of a square is found by multiplying two equal numbers together, the *area* is the *product* of *two equal factors*.

3. The *factors* of a number are other numbers which, multiplied together, produce the given number.

$$\begin{aligned}\text{The factors of } 24 &= 12\cdot 2 \text{ or}\\ &= 8\cdot 3\\ &= 6\cdot 4\\ &= 2\cdot 2\cdot 2\cdot 3\\ &= 2^3\cdot 3\end{aligned}$$

4. A prime number can be exactly divided only by itself and one.

5. Which of the given factors of 24 are prime?

6. Find two factors of 16, 21, 23, 25, 32, 144, 49, 81, $\frac{1}{4}$, $\frac{9}{16}$, 7.

7. Which of these numbers are prime?

8. Find the prime factors of the others.

9. Divide as many as possible into two *equal* factors.

10. (a) We see that 24 has only two different prime factors, 2 and 3. But three 2's are multiplied together with one 3 to give 24. Thus:

$$24 = 2\cdot 2\cdot 2\cdot 3$$

Instead of writing out the factor 2 three times, we may write it once and put a small 3 to the right and a little above it, to show that the 2 must be multiplied by itself 3 times. This small 3 is called the *exponent* of the 2.

$$\begin{aligned} 72 &= 2\cdot 2\cdot 2\cdot 3\cdot 3 \\ &= 2^3\cdot 3^2 \end{aligned}$$

(Read, the cube of two multiplied by the square of three or two cubed times three squared.)

(b) Write the prime factors of the following numbers as shown in 10 (a).

(1)	(2)	(3)	(4)	(5)
12	18	20	24	25
8	27	22	28	39
9	32	14	30	42
16	35	15	36	45
48	54	63	65	75
84	85	96	100	108
125	144	150	200	225
250	175	196	300	400

11. (a) If we are given the area of a square to find its edge, we must find the two equal factors which give this product.

(b) This process is called finding the *square root* of a number.

(c) The sign which tells us to find one of the two equal factors is $\sqrt{\quad}$, written over the number. It is called the *radical sign*.

$\sqrt{25}$ is read "the square root of 25."

12. Given the area of a square to find its edge.

$$\begin{aligned} S_{\square} &= e^2 \\ 25 &= e^2 \\ \sqrt{25} &= \sqrt{e^2} \\ \sqrt{5 \times 5} &= \sqrt{e \times e} \\ 5 &= e \end{aligned}$$

13. 25 is called a *perfect square* because it is exactly 5×5. 15 is not a perfect square because there are no two like factors which multiplied together give exactly 15.

But $\sqrt{15} = 3.87 +$ because $3.87 \times 3.87 = 14.97 +$ which is nearly 15.

Of numbers which are not perfect squares, only the approximate square roots can be found. The degree of approximation depends on the number of decimal places.

$$\begin{aligned} \sqrt{15} &= 3.8\ +, \text{ for } (3.8)^2 = 14.44 \\ \sqrt{15} &= 3.87\ +, \text{ for } (3.87)^2 = 14.9769 \\ \sqrt{15} &= 3.873\ -, \text{ for } (3.873)^2 = 15.000129 \end{aligned}$$

II. Finding Square Root by Prime Factor Method

1. The square roots of numbers may be found in three ways.

The first method of finding the square root is the *prime factor method.* It is used for finding the roots of perfect squares.

(a) What is the square root of 324?

Solution. $\sqrt{324} = ?$

$$\begin{aligned}\sqrt{324} &= \sqrt{4\cdot 81}\\ &= \sqrt{2^2\cdot 3^2\cdot 3^2}\\ &= 2\cdot 3\cdot 3\\ &= 18\end{aligned}$$

Explanation:

We find in 324 there are two 2's and four 3's or two times two 3's. For each pair of equal prime factors in the number, we take one factor for the square root.

2. By the prime factor method find the square root of 144, 441, 1225.

3. Find the value of $\sqrt{729}$, $\sqrt{196}$, $\sqrt{484}$, $\sqrt{625}$.

4. If a square field contains 2025 sq. rd., what is the length of each side? How many rods of fencing are needed?

III. Finding Square Root by Mechanical Method

1. (a) The second method for finding the square root of a number is the *mechanical method.*

(b) Before giving the process, let us examine the squares of a few numbers.

$1^2 =$	1	Numbers with units only have how many digits in their squares?
$3^2 =$	9	
$5^2 =$	25	Numbers with two figures or tens and units have how many digits in their squares?
$9^2 =$	81	
$10^2 =$	100	
$15^2 =$	225	Since $1 = 1^2$ and $100 = 10^2$, the square root of a number between 1 and 100 must be a number between 1 and 10.
$55^2 =$	3,025	
$90^2 =$	8,100	
$99^2 =$	9,801	Since $100 = 10^2$ and $10{,}000 = 100^2$ the square root of a number between 100 and 10,000 must be a number between 10 and 100.
$100^2 =$	10,000	
$999^2 =$	998,001	
$1000^2 =$	1,000,000	

(c) How many digits will be in the square root of a number expressed by two figures?

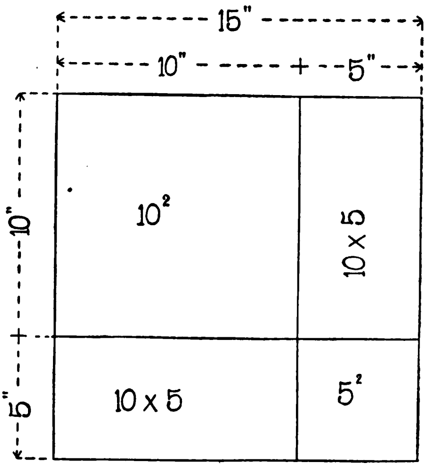
(d) How many figures will be in the square root of a number expressed by three or four figures?

(e) How many digits are there in the square root of 4225? Of 256? Begin at units and point off groups or periods of two figures each; thus 42′ 25 or $\overline{42}\ \overline{25}$; 2′ 56 or $\overline{2}\ \overline{56}$.

The left-hand period may have only one figure. For *each* group or period there will be *one* figure in the square root.

(f) How many figures are in the square root of 3969; 441; 529; 49; 15625?

(g) On squared paper, draw a 15-inch square, using $\frac{1}{8}$ inch to represent an inch.



Since 15 = 10 + 5, the square may be divided as in the figure.

How many smaller squares does the large square contain? how many rectangles?

What is the area of each?

From the figure we see that

$$\begin{aligned} 15^2 &= 10^2 + 2(10 \times 5) + 5^2 \\ &= 100 + 100 + 25 \\ &= 225 \end{aligned}$$

By multiplication we see the same truth.

$$
\begin{array}{rrrrr}
15 = & & 10 & + & 5 \\
\underline{15} = & & \underline{10} & \underline{+} & \underline{5} \\
 & & (10 \times 5) & + & 5^2 \\
 & 10^2 & + (10 \times 5) & & \\
\hline
15^2 = & 10^2 & + 2(10 \times 5) & + & 5^2 \\
= & 100 & + 100 & & + 25 \\
= & 225 & & &
\end{array}
$$

or

$$
\begin{array}{rrrr}
 & \text{tens} & + & \text{units} \\
 & \underline{\text{tens}} & \underline{+} & \underline{\text{units}} \\
 & (\text{tens} \times \text{units}) & + & (\text{units})^2 \\
(\text{tens})^2 + & (\text{tens} \times \text{units}) & & \\
\hline
(\text{tens})^2 & + 2(\text{tens} \times \text{units}) & + & (\text{units})^2
\end{array}
$$

This statement written as a formula is:

$$(t + u)^2 = t^2 + 2(t \times u) + u^2$$

(h) Use the formula to find the square of 35.

Solution.

$$
\begin{array}{rcl}
35 & = & 30 + 5 \\
(t + u)^2 & = & t^2 + 2(t \times u) + u^2 \\
(30 + 5)^2 & = & 30^2 + 2(30 \times 5) + 5^2 \\
 & = & 900 + 300 + 25 \\
 & = & 1225
\end{array}
$$

Therefore $35^2 = 1225$

(i) Use this formula to find the square of 16, 22, 14, 17, 13, 25, 61.

(j) This study about squaring numbers will help to make clear the mechanical method of finding square root.

2. The *Mechanical Method* for Finding the Square Root of a Number.

(a) Find the square root of 225; of 3969.

Process

$$
\begin{array}{rr|l}
 & 2'25 & \underline{15} \\
 & \underline{1} & \\
\text{T.D} = 2 & 125 & \\
\text{C.D} = 25 & \underline{125} & \\
\end{array}
$$

$\sqrt{225} = 15$

$$
\begin{array}{rr|l}
 & 39'69 & \underline{63} \\
 & \underline{36} & \\
\text{T.D} = 12 & 369 & \\
\text{C.D} = 123 & \underline{369} & \\
\end{array}
$$

$\sqrt{3969} = 63$

Explanation of $\sqrt{3969}$

Begin with unit's digit and point off groups of two digits each. (Why?)

The largest square in the left-hand period is 36 or 6^2. Therefore 6 is the first figure in the root.

Subtract 6^2 or 36 from 39, the left-hand period, leaving 3. Bring down the next period, making 369.

For a trial divisor (T.D.), *double* that part of the root already found. That is, double 6, the tens digit, which gives

12 for a trial divisor. (Why must you double the tens digit?) In using the trial divisor, leave off the last figure of the dividend or 9. Try 12 into 36 or 3. Put 3 in the root and annex it to 12, making the complete divisor (C.D.) 123.

Multiply the complete divisor by the last figure in the root. Continue this process until all periods are used.

(b) A square field contains 169 sq. rd. How long is the side?

(c) Find the value of:

(1)	$\sqrt{841}$	(2)	$\sqrt{1024}$	(3)	$\sqrt{1444}$	(4)	$\sqrt{1681}$
(5)	$\sqrt{2401}$	(6)	$\sqrt{2209}$	(7)	$\sqrt{361}$	(8)	$\sqrt{289}$
(9)	$\sqrt{2304}$	(10)	$\sqrt{529}$	(11)	$\sqrt{1369}$	(12)	$\sqrt{3136}$
(13)	$\sqrt{4489}$	(14)	$\sqrt{6561}$	(15)	$\sqrt{15,129}$	(16)	$\sqrt{46,225}$
(17)	$\sqrt{3025}$	(18)	$\sqrt{4356}$	(19)	$\sqrt{3721}$	(20)	$\sqrt{6889}$
(21)	$\sqrt{55,225}$	(22)	$\sqrt{7056}$	(23)	$\sqrt{4096}$	(24)	$\sqrt{225,625}$
(25)	$\sqrt{643,204}$	(26)	$\sqrt{95,481}$	(27)	$\sqrt{258,064}$	(28)	$\sqrt{494,209}$

(d) Find the square roots of the following:

(1)	5184	(2)	132,496	(3)	2116	(4)	1764
(5)	7921	(6)	3249	(7)	2304	(8)	3481
(9)	9216	(10)	6241	(11)	1296	(12)	2809
(13)	6724	(14)	961	(15)	1156	(16)	123,904
(17)	9604	(18)	6561	(19)	62,500	(20)	10,609
(21)	42,025	(22)	11,881	(23)	474,721	(24)	552,049
(25)	91,204	(26)	160,801	(27)	110,889	(28)	8464
(29)	7396	(30)	402,849	(31)	368,449	(32)	622,521

3. Square Root of Decimal Fractions.

$$\left(\tfrac{1}{10}\right)^2 = \tfrac{1}{100} \text{ or } (.1)^2 = .01$$
$$\left(\tfrac{1}{100}\right)^2 = \tfrac{1}{10,000} \text{ or } (.01)^2 = .0001$$
$$\left(\tfrac{1}{1000}\right)^2 = \tfrac{1}{1,000,000} \text{ or } (.001)^2 = .000001$$

(a) Just as with whole numbers, the square root has one half as many decimal places as the square. So we must point off periods of two figures, beginning at the decimal point.

Each decimal period must have two figures. If necessary, annex ciphers. .115 is the same as .1150, so the $\sqrt{.115}$ is the same as $\sqrt{.1150}$.

With a mixed decimal, begin at the decimal point and point off periods to the right and left.

(b) Illustrations:

(1) $\sqrt{1.5625} = 1.25$

$$
\begin{array}{r|l}
 & \overline{1}.\ \overline{56}\ \overline{25}\ \ \underline{1.25} \\
 & 1 \\
\hline
22 & 56 \\
 & 44 \\
\hline
245 & 1225 \\
 & 1225 \\
\hline
\end{array}
$$

(2) $\sqrt{3} = \sqrt{3.000000} = 1.732 +$

$$
\begin{array}{r|l}
 & \overline{3}.\ \overline{00}\ \overline{00}\ \overline{00}\ \ \underline{1.732+} \\
 & 1 \\
\hline
27 & 200 \\
 & 189 \\
\hline
343 & 1100 \\
 & 1029 \\
\hline
3462 & 7100 \\
 & 6924 \\
\hline
\end{array}
$$

(c) Find the value of:

(1) $\sqrt{52.2729}$, $\sqrt{3}$, $\sqrt{2}$, $\sqrt{.2}$

(2) $\sqrt{60.3729}$, $\sqrt{5}$, $\sqrt{.5}$, $\sqrt{.06}$

(3) $\sqrt{73.96}$, $\sqrt{.025}$, $\sqrt{.016}$, $\sqrt{3.143}$

4. The Square Root of Common Fractions.

$$\left(\tfrac{3}{4}\right)^2 = \tfrac{9}{16}$$

Therefore $\sqrt{\tfrac{9}{16}} = \tfrac{3}{4}$

(a) If both terms of a fraction are perfect squares, to find its square root what two roots must be found?

(b) Find the value of $\sqrt{\frac{1}{4}}$, $\sqrt{\frac{4}{9}}$, $\sqrt{\frac{25}{64}}$, $\sqrt{\frac{23}{36}}$, $\sqrt{\frac{1}{25}}$.

(c) Is the square of 2 greater or less than 2?
Is the square of $\frac{1}{2}$ greater or less than $\frac{1}{2}$?

(d) Is the square root of 25 greater or less than 25?
Is the square root of $\frac{1}{25}$ greater or less than $\frac{1}{25}$?

(e) Find the square root of $\frac{2}{3}$.

We see that some other way must be found to find this root, because 2 and 3 are not perfect squares. We may choose one of two ways. We may reduce such a fraction to the decimal form and take the root, or we may make the denominator a perfect square and solve it thus; because $\frac{2}{3} = \frac{6}{9}$.

$$\sqrt{\tfrac{2}{3}} = \sqrt{\tfrac{2}{3} \times \tfrac{3}{3}} = \sqrt{\tfrac{6}{3^2}} = \sqrt{\tfrac{1}{3^2}} \times \sqrt{6} = \tfrac{1}{3}\sqrt{6}$$
$$= \tfrac{1}{3} \times 2.449$$
$$= .816$$

(f) Another illustration:

$$\sqrt{\frac{5}{8}} = \sqrt{\frac{5}{2^2 . 2} \times \frac{2}{2}} = \sqrt{\frac{10}{2^2 . 2^2}}$$
$$= \tfrac{1}{2} \times \tfrac{1}{2}\sqrt{10}$$
$$= \tfrac{1}{4}\sqrt{10}$$
$$= \tfrac{1}{4} \times 3.162 +$$
$$= .7905 +$$

(g) Decimal method·

$\frac{5}{8} = .625$

$$\begin{array}{r|l}
 & \overline{.62}\ \overline{50}\ \overline{00}\ \overline{00} \qquad \underline{.7905 +} \\
 & \underline{49} \\
149 & 1350 \\
 & \underline{1341} \\
 & 90{,}000 \\
15{,}805 & \underline{79{,}025}
\end{array}$$

Note: The value of the square roots of such numbers as 2, 3, 5, 6, 7, 10 ... are kept in convenient tables by those who use them a great deal. For such persons the method given in (f) is shorter than the decimal method.

(h) Find the value of $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{1}{6}}$, $\sqrt{\frac{5}{6}}$, $\sqrt{\frac{2}{5}}$, $\sqrt{\frac{3}{8}}$, $\sqrt{\frac{3}{5}}$, $\sqrt{\frac{7}{8}}$.

5. Changing Common Fractions to Decimal Forms.

(a) We learned that, to find the square root of most fractions, they must be reduced to decimal form. By division, we discovered that some fractions can be changed to decimal form exactly or without remainder. Such fractions are called *terminating decimals* (decimals with an end). Others do not terminate, no matter how far one carries the division.

(b) By inspection, one can tell whether or not the fraction is terminating, and thus save the time of carrying out lengthy division in the hope that it will finally come out "even" or without remainder. Decimal fractions are those having ten or a power of ten as their denominators; as $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$. These may be written as

$$\frac{1}{2\cdot 5},\ \frac{1}{2^2\cdot 5^2},\ \frac{1}{2^3\cdot 5^3},\ \frac{1}{2^4\cdot 5^4}$$

We see, therefore, that every decimal must have only 2's and 5's as factors in the denominator, and that the number of 2's and 5's must be the same.

(c) If any fraction in its lowest terms has any factor other than 2 or 5 in its denominator, it can *never* be reduced to a terminating decimal; for example, $\frac{5}{12} = \frac{5}{2^2\cdot 3}$. Since one cannot get rid of the factor 3 in the denominator, one should know by inspection that it can never be changed into a terminating decimal. The division should be carried to as many decimal places as needed, perhaps three or four.

$$\frac{5}{12} = .4166+$$

$$\begin{array}{r|l} & .4166+ \\ \hline 12 & 5.0000 \end{array}$$

(d) Every fraction, which in its lowest terms has only 2's and 5's in its denominator, may be easily changed to

decimal form by making the number of 2's and 5's equal. Thus

$$\frac{3}{40}=\frac{3}{5\cdot 2^3}=\frac{3}{5\cdot 2^3}\times\frac{5^2}{5^2}=\frac{75}{5^3\cdot 2^3}=.075$$

Explanation:

Factor the denominator 40 and write the fraction as $\frac{3}{5\cdot 2^3}$.

Since there are three 2's and only one 5, we must multiply both terms by 5^2 which gives $\frac{75}{5^3\cdot 2^3}$.

$5^3\cdot 2^3 = 10^3$ or 1000, therefore three decimal places are necessary. The decimal form is always the numerator and the number of decimal places is the number of factors 2 or 5 in the denominator. In other words, the number of decimal places equals the exponent of the factors of the denominator.

(e) Other examples:

$$\frac{3}{16}=\frac{3}{2^4}=\frac{3\cdot 5^4}{2^4\cdot 5^4}=.1875$$

$$\frac{7}{80}=\frac{7}{2^4\cdot 5}=\frac{7\cdot 5^3}{2^4\cdot 5\cdot 5^3}=.0875$$

(f) Tell by inspection which of the following fractions can be changed to terminating decimals. Change them by the factor method.

$\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{18}$, $\frac{7}{20}$, $\frac{1}{32}$, $\frac{7}{8}$, $\frac{1}{12}$, $\frac{7}{16}$, $\frac{7}{25}$, $\frac{5}{6}$, $\frac{8}{9}$

IV. Finding Square Root by Inspection

The third method of finding square root is by inspection. Its use necessitates one's knowing the squares of all numbers from 1 to 30. The method is explained in the supplement.

V. Review Problems

1. Find the square root of:

(a)	4761	(e)	1960
(b)	585,225	(f)	2250
(c)	4624	(g)	5335
(d)	6255	(h)	256,036

2. Find the value of the following:

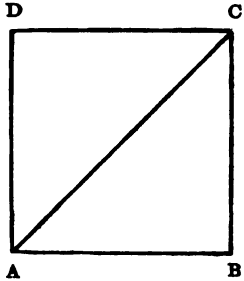
(a)	$\sqrt{\frac{2}{3}}$	(c)	$\sqrt{\frac{3}{20}}$	(e)	$\sqrt{\frac{1}{32}}$	(g)	$\sqrt{\frac{5}{18}}$
(b)	$\sqrt{\frac{1}{12}}$	(d)	$\sqrt{\frac{1}{50}}$	(f)	$\sqrt{\frac{5}{12}}$	(h)	$\sqrt{\frac{1}{24}}$

CHAPTER THREE

A STUDY OF TRIANGLES

I. Reading of Angles

1. Cut out a square and carefully fold it between two opposite corners.



2. The name of the line made by the fold is *diagonal.*

Diagonal comes from two Greek words which mean *through the angle.*

3. Into what kind of figures is the square divided by the diagonal?

4. The word *triangle* means *three-cornered.* We write it thus, △; plural ▲.

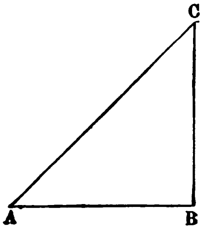
5. (a) Do the two parts of the square exactly fit each other?

(b) What part of the area of the square is in the triangle?

(c) If the side of the square is 4, what is its area?

(d) What is the area of each triangle?

6. (a) How does the diagonal divide the angles at the corners of the square at A and C?



(b) How big is each angle of the triangle?

7. This figure may be named the triangle ABC, ($\triangle ABC$).

Each corner is the *vertex* of the angle made by the lines that meet there.

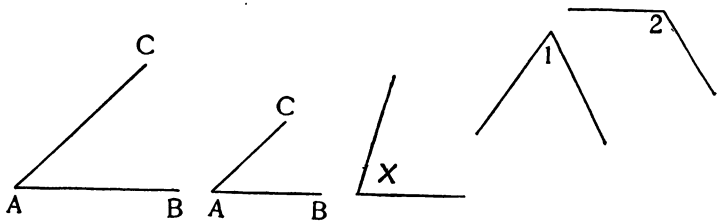
8. How to read the names of angles.

(a) Sometimes we name an angle with three letters, reading from one end of a side to the vertex, then to the other side.

(1) In the triangle, the $\angle$ at the vertex A is read $\angle BAC$.

In the triangle, the $\angle$ at the vertex B is read $\angle ABC$

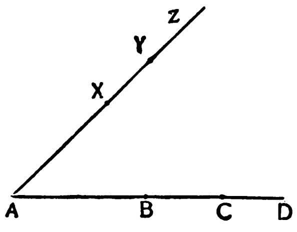
In the triangle, the $\angle$ at the vertex C is read $\angle ACB$.



(2) In the name of the angle, where is the vertex letter always found?

(b) Sometimes a small letter or number is placed within the angle close to the vertex; as, $\angle x$, $\angle 1$, $\angle 2$.

9. The size of the angle depends upon the opening between the sides and not upon the length of the lines.



The $\angle A$ may be read

$\angle BAX$, $\angle BAY$, $\angle BAZ$
$\angle CAX$, $\angle CAY$, $\angle CAZ$
$\angle DAX$, $\angle DAY$, $\angle DAZ$

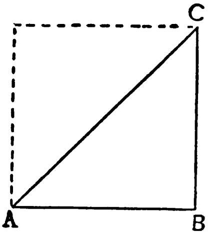
Using different points on the sides of the angle does not change its size.

II. The Sum of the Angles of a Triangle

1. Since $\angle B$ is an angle of a square

$$\angle B = 90^\circ \text{ or a rt. } \angle.$$

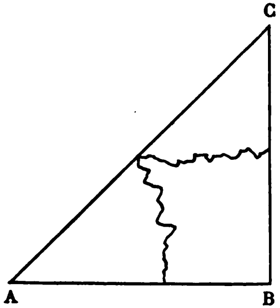
2. $\angle A = \angle C = 45^\circ = \frac{1}{2}$ rt. $\angle$. Why?

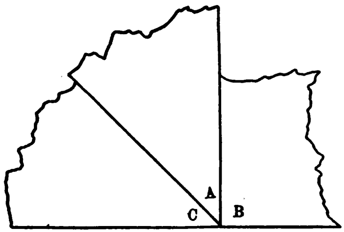


3. What is the sum of $\angle s\ A$, B, and C?

$$\begin{aligned} \angle B &= 90^\circ = 1 \text{ rt. } \angle. \\ \angle A &= 45^\circ = \tfrac{1}{2} \text{ rt. } \angle. \\ \angle C &= 45^\circ = \tfrac{1}{2} \text{ rt. } \angle. \\ \hline \angle B + \angle A + \angle C &= 180^\circ = 2 \text{ rt. } \angle s. \end{aligned}$$

4. Cut off the three corners of the triangle and place them carefully about the point of your folded $\angle$ paper, thus:

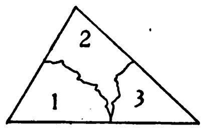


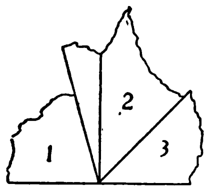


What does this show?

5. If the three corners are tinted with different colored crayons, the parts may be seen more distinctly.

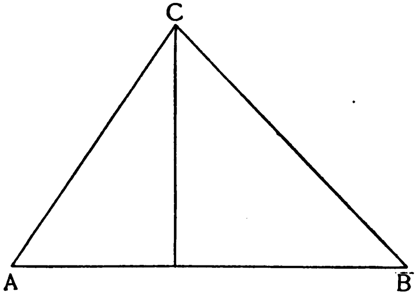
6. Cut out many differently shaped triangles and test the sums of the angles in the same way.

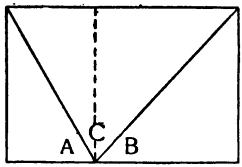




7. Use your protractor to measure the three angles of several triangles and see if their sum is 180°.

8. What is your conclusion as to the sum of the angles of any triangle?





(a) Draw $\triangle ABC$.

(b) Draw a perpendicular from the vertex of the largest angle.

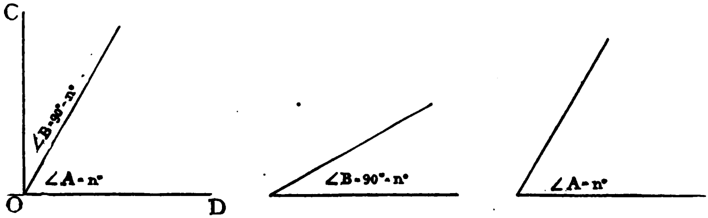
(c) Cut out the triangle and fold over the corners so as to meet at the foot of the perpendicular.

(d) What does this prove?

III. Complementary Angles

1. What is the sum of the two small angles of a right triangle?

2. Any two angles whose sum is a right angle or 90° are *complementary* angles. Each angle is the *complement* of the other.



3. Angles A and B are complementary angles because their sum is the rt. $\angle COD$.

4. If $\angle A = 65°$, then $\angle B = 90° - 65° = 25°$.

If $\angle A = n$ degrees, how many degrees are in $\angle B$?

5. With your protractor draw the following angles. Compute and draw the complement of each.

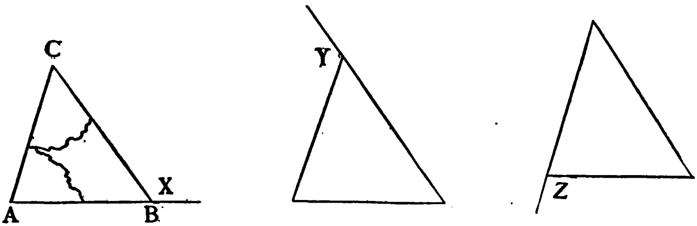
(a) $30°$	(f) $68°$	(k) $19°$	(p) $\frac{1}{2}$ rt. $\angle$.
(b) $23°$	(g) $5°$	(l) $35°$	(q) $\frac{1}{3}$ rt. $\angle$.
(c) $45°$	(h) $85°$	(m) $80°$	(r) $\frac{3}{5}$ rt. $\angle$.
(d) $75°$	(i) $60°$	(n) $6°$	(s) $\frac{3}{4}$ rt. $\angle$.
(e) $12\frac{1}{2}°$	(j) $2\frac{1}{2}°$	(o) $90°$	(t) $\frac{1}{6}$ rt. $\angle$.

IV. Interior and Exterior Angles of Polygons

1. We call angles *within* a triangle or parallelogram, *interior* angles to distinguish them from *exterior* angles which are *outside* the figure.

2. Exterior angles of triangles.

(a) To form an *exterior angle* of a triangle or other polygon extend any side in one direction only. The angle formed by one side of the triangle and an adjacent side extended is an *exterior angle.*



(b) $\angle$s X, Y, and Z are all *exterior* $\angle$s.
Can you make others in these triangles?

(c) Draw a figure as $\triangle ABC$. Cut out the $\angle$s at A and C and carefully fit them into the exterior $\angle X$. Do this with several triangles, or until you feel sure that:

An *exterior* angle at one vertex of a triangle is exactly equal to *sum* of the two *interior* angles at the *other* vertices.

(d) Test this by measuring the $\angle$s with your protractor.

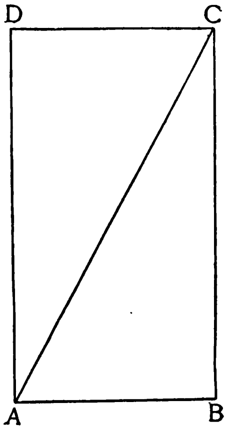
B. TRIANGLES FROM RECTANGLES

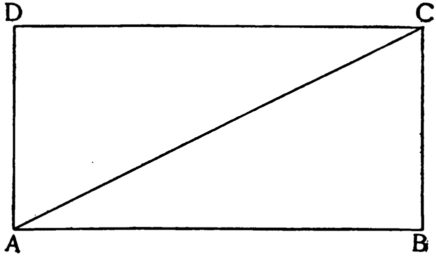
1. You have found that the diagonal of a square divides it into two equal triangles.

2. Draw and cut out a rectangle. Fold on one diagonal. Do the two parts fit?

3. Cut along the diagonal. Now can you make the two parts fit?

4. What part of the rectangle is now in each $\triangle$?





5. (a) In our first formula for the area of a rectangle we used the names length and width because they were the names for the dimensions of the oblong block. We have other names. The side on which the rectangle stands is the base, as the line *AB*. The side perpendicular to the base is called the height or altitude, as the lines *BC* or *AD*.

(b) We found that the area of
the surface of a rectangle = length $\times$ width, or,
the surface of a rectangle = base $\times$ height.

Therefore we may write the formula thus:

$$S_{\square} = bh$$

C. CLASSIFICATION OF TRIANGLES

I. According to Size of Angles

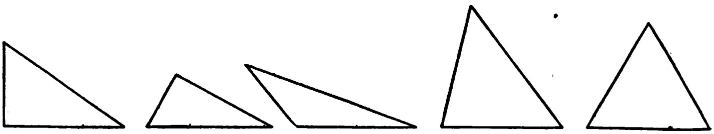
Triangles have different names according to the size of their angles.

1. A triangle having one angle a right angle is a *right triangle.*

2. A triangle having one angle an obtuse angle is an *obtuse triangle.*

3. A triangle having *all* angles acute is an *acute triangle.*

An acute triangle having all of its angles equal is called an *equiangular triangle.* Why?

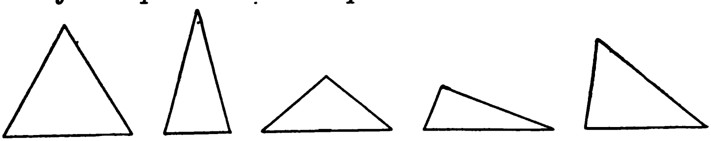


Right Obtuse Acute Equiangular

II. According to Equality of Sides

Triangles have different names also according to the number of the sides that are equal.

1. A triangle having its 3 sides equal is an *equilateral triangle. Equilateral* means *equal sided.*



Equilateral Isosceles Scalene

2. A triangle having 2 of its sides equal is an *isosceles triangle. Isosceles* means *equal legs.*

3. A triangle having no sides equal is a *scalene triangle. Scalene* means *uneven.*

4. We seldom use the word scalene; but when we mean either of the two special kinds, we are very careful to specify them by name.

5. (a) Draw one or more of each kind of triangles as accurately as possible.
   (b) Label each kind.
   (c) Measure the angles in each.
   (d) Find the sum of the angles in each triangle.

D. AREAS OF TRIANGLES

I. Area of Right Triangle

1. We have seen that the triangles formed by the diagonal of a rectangle are two equal right triangles.

2. Those formed by the diagonal of a square are two equal isosceles right triangles.

3. Since $S_{\square} = bh$

and since the rt. $\triangle = \frac{1}{2} \square$

therefore, S rt. $\triangle = \frac{1}{2} bh$

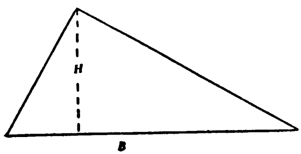
4. Which sides of the right $\triangle$ may be the base and height?

5. The two $\perp$ sides are called the legs of the rt. $\triangle$.

6. The longest side, the one opposite the right angle, is called the *hypotenuse*.

7. *Hypotenuse* means *stretched under*. Which angle is it stretched under?

8. The hypotenuse may be considered the base of a right triangle. In that case, the altitude or height is a line drawn from the vertex of the rt. angle $\perp$ the hypotenuse.



9. How to find the altitude of any triangle (except certain altitudes of obtuse $\triangle$s).

(a) By folding.

(1) Draw and cut out a triangle. Make a fold from one vertex to the opposite side, so that

the two parts of that side fall exactly together.

Unfold and measure the ∠s on either side of the fold with your protractor or ∠-paper. This fold is one of the altitudes of the △.

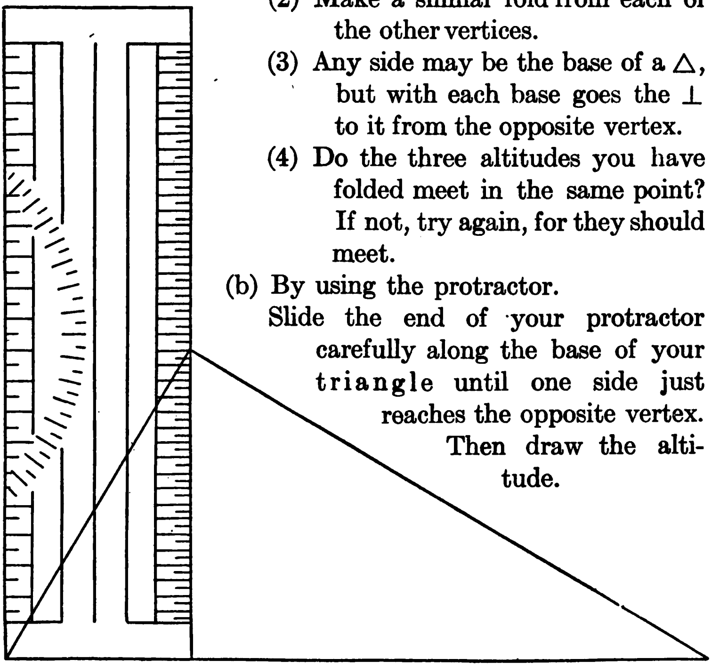
(2) Make a similar fold from each of the other vertices.

(3) Any side may be the base of a △, but with each base goes the ⊥ to it from the opposite vertex.

(4) Do the three altitudes you have folded meet in the same point? If not, try again, for they should meet.

(b) By using the protractor.

Slide the end of your protractor carefully along the base of your triangle until one side just reaches the opposite vertex. Then draw the altitude.



(c) More exact constructions with a compass will be shown in Chapter Four.

II. Area of Scalene Triangle.

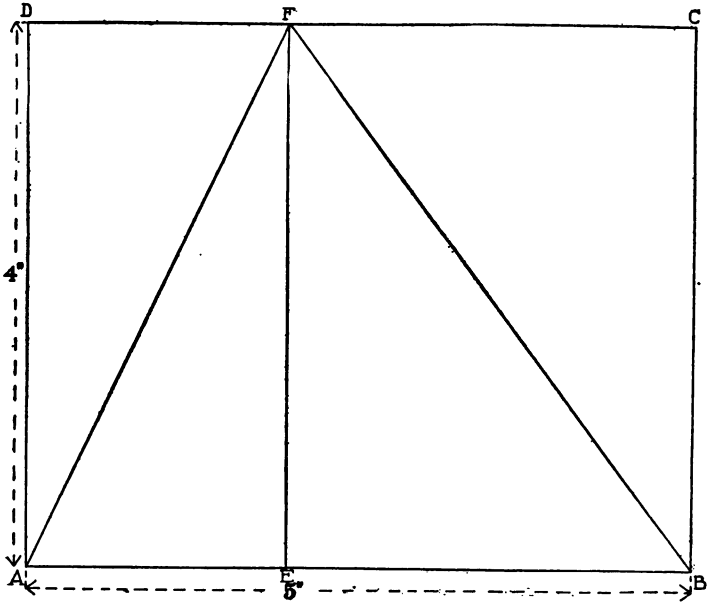
1. (a) On squared paper, draw a rectangle whose base is 5 inches and whose height is 4 inches. Plain paper may be used, but you must be very careful to make exact right angles.

(b) At E, any point in the base not the mid point, draw $EF \perp AB$. Use protractor or folded paper to make the rt. $\angle$ if plain paper is used.

(c) Draw the diagonals AF and BF.

(d) $ABCD$ is a $\square$, with base AB and height EF.

(e) ABF is a $\triangle$, with base AB and height EF.



(f) Cut out the $\square$ $ABCD$ and cut along AF and BF.

(g) Fit $\triangle$ ADF on $\triangle$ AEF and
fit $\triangle$ BCF on $\triangle$ BEF.

(h) We see then that $\triangle\ ABF = \frac{1}{2} \square\ ABCD$.

But, $S \square\ ABCD = bh = 5 \times 4 = 20$ **sq. in.**

Therefore $S \triangle\ ABF = \frac{1}{2} bh = \frac{1}{2} \times 5 \times 4 = 10$ **sq. in.**

(i) Since ABF is a scalene $\triangle$, the formula,

$$S\triangle = \frac{1}{2}\,bh, \text{ must be true for any triangle.}$$

Translate this formula into an English statement.

(j) Check the area by counting the small squares or equal parts of squares on the squared paper.

(k) Test this formula with other figures.

2. Exercises.

(a) Draw three altitudes in a triangle. Measure each base and altitude to the nearest tenth of a centimeter. Find the area from the three sets of measurements. Are they approximately equal?

(b) Find the area of a triangle whose base is $6'\,4''$ and whose height is $4'\,6''$.

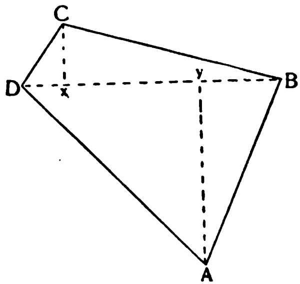
(c) Find the area of the following triangles, using the formula,

$$S\triangle = \frac{1}{2}bh$$

Estimate the areas first:

	b	h	S
(1)	$9''$	$6''$	
(2)	$4\frac{3}{4}''$	$2\frac{1}{2}''$	
(3)	$5'\,8''$	$3'\,4''$	
(4)	$6.6'$	$3.5'$	
(5)	120 rd.	62.5 rds.	
(6)	$17\frac{1}{2}''$	$13\frac{3}{4}''$	
(7)	$3'\,9''$	$2'\,3''$	
(8)	$2'$	$18''$	

(d) A farmer has an irregularly shaped field $ABCD$. By dividing it into two triangles, he can measure each to find the area of the field.



What three lines must he measure?

If $BD = 280$ feet, $CX = 75$ feet, and $AY = 190$ feet, find the area of the field.

CHAPTER FOUR

CONSTRUCTIONS

A. CONSTRUCTIONS OF TRIANGLES

1. On squared paper it is easy to draw squares, rectangles, and other figures correctly, but with a ruler and compass we can draw exact figures of all kinds on plain paper.

Before experimenting any further with measurements of different figures we shall learn to construct them with our instruments.

Some suggestions follow:

(a) Always keep pencils well sharpened. A pencil sharpener should be in every mathematics room.

(b) Be very exact in all measurements.

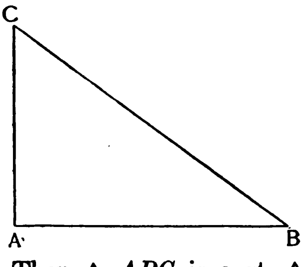
(c) Keep work neat.

(d) One cannot be too painstaking in all construction work.

(e) Keep all figures of reasonable size, neither too large nor too small.

(f) Do not lift compass point any more than necessary.

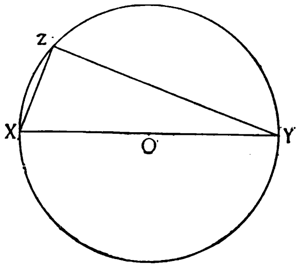
2. *To construct a right triangle.*



(a) Draw a line AB any length.
With your protractor draw a line $AC \perp AB$ at point A. Make AC any length. Draw BC. Then $\triangle ABC$ is a rt. $\triangle$.

(b) With your compass draw a circle.
(The sign for circle is $\odot$.)
Through the center O, made by the point of the compass, draw a line cutting the circle in two parts.

(This line is a diameter. It means the *measure through*, that is, *through* the center.)



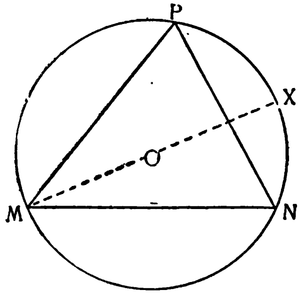
Letter this diameter XY. Take any other point on the ⊙ as Z, and draw XZ and YZ. With your protractor measure the $\angle Z$. What kind of a △ is XYZ? Why?

NOTE: The $\angle Z$ is said to be inscribed in a semicircle, because its vertex lies on the circle and its sides pass through the ends of the diameter.

The statement that "every angle that is inscribed in a semicircle is a right angle" was proved to be true by a famous Greek over 2500 years ago. The man was Thales, who was born about 640 B.C.

At the time Thales lived, there was no such thing as arithmetic or algebra, and only a very few facts of geometry were known. His discovery of the truth about the right angle being inscribed in a semicircle was the cause of a great celebration, and it is said that he sacrificed an ox to the immortal gods.

When you study geometry in the senior high school, you may learn how Thales proved this theorem.



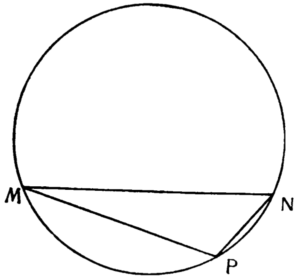
3. *To construct an acute triangle.*

(a) With your protractor, draw a triangle with angles less than 90°.

(b) In a circle, construct an acute triangle as follows: Draw a circle and a line not a diameter, as MN. Draw a diameter MX. On the upper part of the ⊙ between M and X take some point near enough

to X so that when PM and PN are drawn, the center O will be within the △.

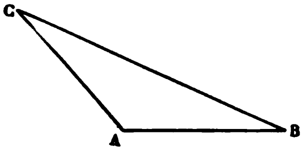
With your protractor measure all the ∠s of the △ PMN. What kind of △ is it? Why?



4. *To construct an obtuse triangle.*

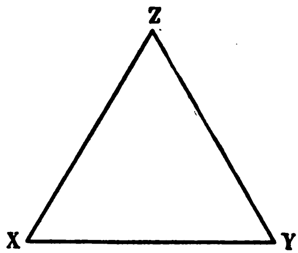
(a) Draw a ⊙ and the line MN as in exercise 3. Take P any point in the lower part of the ⊙. Draw PM and PN. With your protractor measure ∠P. What kind of △ is MNP? Why?

(b) With your protractor make an obtuse ∠ at any point A. Draw a line cutting both sides of ∠A. What kind of △ is ABC? Why?



5. (a) The curved line which makes a circle is sometimes called its circumference. Any small part of a *circumference* is called an *arc*. We write it thus, ⌒, ⌒s.

(b) The point where two lines meet or cut each other is called their *point of intersection* or their *intersection*. *Intersect* means *to cut into*. We speak of two intersecting streets, meaning two streets that cross each other.



6. *To construct an equilateral triangle.*

(a) Draw a line XY as long as you want the side of

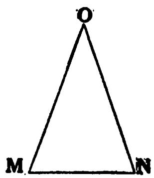
your $\triangle$. Measure the line XY between the points of your compass.

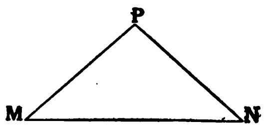
With the sharp point on X make an arc above the line. Then put the sharp point on Y, and without changing your compass make an arc that will intersect the first one, as at Z. Draw XZ and YZ.

(b) With your ruler measure the length of the three sides of the $\triangle$. Measure them with your compass. With which one can you measure more exactly? What kind of $\triangle$ is XYZ? Why?

With your protractor measure the three $\angle$s. What other name can you give? How many degrees are in each $\angle$ of an equilateral $\triangle$?

7. *To construct an isosceles triangle.*





(a) Draw a line MN as long as you want the side of your $\triangle$. With your compass measuring *more* than MN draw arcs as before, intersecting at O. Draw OM and ON as in Fig. A.

In Fig. B draw the arcs with a measure *less* than MN. What kind of $\triangle$s are MON and MPN?

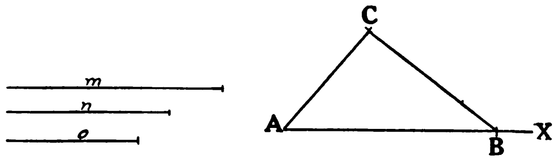
(b) Is every equilateral $\triangle$ isosceles? Is every isosceles $\triangle$ equilateral?

(c) With your protractor measure the $\angle$s M and N. Make a number of isosceles angles and measure the angles at each end of the base. These angles are called *base angles*.

(d) What have you discovered about the two base $\measuredangle$s of an isosceles $\triangle$?

(e) The other angle, i.e. the angle opposite the base, is called the *vertex angle.*

8. *To construct a scalene triangle.*



Draw three lines of different lengths, as m, n, o. On any other line, as AX, with your compass measure $AB = m$. With your compass measure n, and with the sharp point at A make an arc.

Likewise with the measure o and the point at B make an arc intersecting the other arc at C. Draw AC and BC. What kind of triangle is ABC? Why?

B. CONSTRUCTION OF LINES

I. To Draw Perpendicular Lines

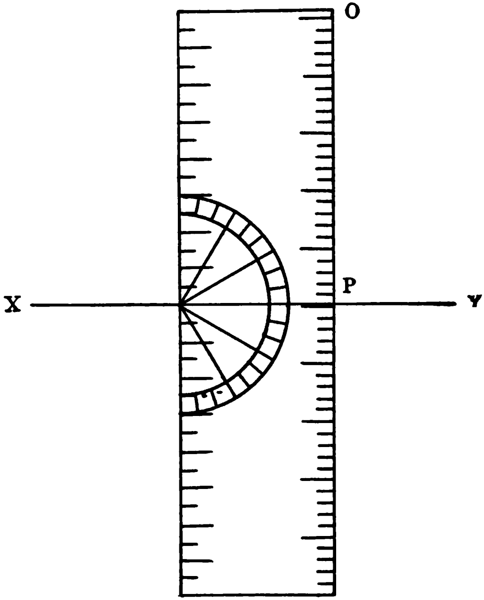
1. *To draw a line perpendicular to another line at a given point in the line.*

(a) By using a protractor.

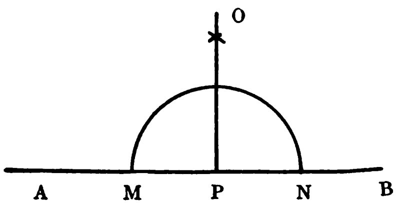
Suppose P is the point in the given line XY at which a $\perp$ is to be erected.

Place the protractor so that the 90° angle line coincides with the line XY. Slide the protractor until its edge just touches point P, being careful to keep the two lines exactly together. Draw a line through P along the edge of the protractor, as OP. The line OP is perpendicular to the line XY at P.

For yourself find another way of using your protractor to draw a perpendicular to a given line at a given point.



(b) By using a compass.

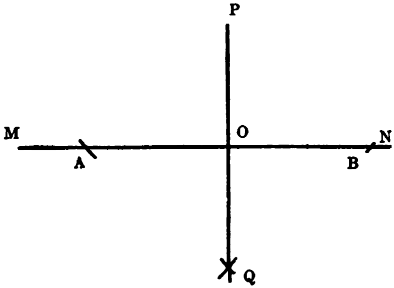


Suppose P is the point in the line AB at which the $\perp$ is to be erected. Place the sharp point of the compass at P and draw a semicircle (half circle) which cuts AB at M and N, or draw two arcs cutting AB at M and N. Open the compass a little more, put the sharp point first at M and draw an arc above P, and then at N and draw an arc intersecting the first one at O. Draw a line from

O to P. OP is $\perp$ to AB. Measure the $\angle APO$ and $\angle BPO$ with your protractor.

What does the size of these $\measuredangle$s prove about the line OP?

2. *To draw a line perpendicular to another line from a point* **not** *in the line.*



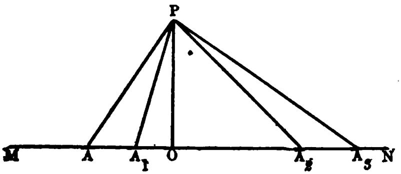
(a) Suppose P is the point from which a $\perp$ is to be drawn to MN. Place the sharp point of the compass on P, and with an opening great enough, draw two arcs cutting the line MN in two points, as A and B.

Put the sharp point of the compass on A and draw an arc below P, then on B, and draw an arc cutting the first at Q. Draw a line from P to Q. Call the point at which PQ cuts MN the point O.

(b) Measure the $\measuredangle$s NOP and MOP.

What do these measures show?

(c) From P draw several lines to MN.



PA_1 is read PA sub 1. PA_2 is read PA sub 2. The same kind of lettering with different numbers written below the letters shows that the lines were drawn in the same way but at different points. How are all the lines PA, PA_1, PA_2, PA_3 drawn? Are these lines longer or shorter than the $\perp$ PO?

(d) Measure the angle that each of these lines makes with MN. How do they compare with $\angle PON$?
As the point A moves away from O, what happens to the size of the $\angle$?
How many lines can you draw from $P \perp$ to MN?

(e) What is the shortest line that you can draw from a point to a given line?

(f) The *distance* from a point to a line always means the shortest distance.

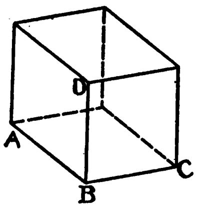
(g) What is the distance of P from MN?

(h) Do you believe the following?

(1) The shortest distance from a point to a line is the perpendicular drawn from the point to the line.

(2) One and only one perpendicular can be drawn to a line at the same point.

(3) If you believe the above statement No. (2), examine your cube or oblong.



How many lines are $\perp$ to DB at point B?
Can there be more than two lines $\perp$ to DB at B?
How many can there be, if we consider only one face or plane surface at a time?
Would statement No. (2) be correct if we limit it to lines in the same plane?

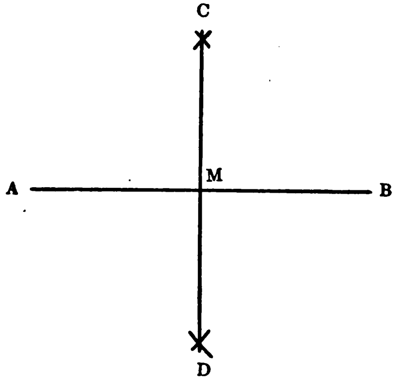
In the same plane, one and only one perpendicular can be drawn to a line at the same point.

(4) Is this statement true?

In the same plane, one and only one perpendicular can be drawn from a given point to a line.

II. To Draw Lines of Bisection

1. *To bisect a line.*



(a) Suppose AB is the line to be bisected (cut in two equal parts).

Open the compass a little more than half of the line AB. Put the sharp point on A and draw one arc above and another below the line. Then from B make two arcs intersecting the others, being careful not to change the opening of the compass. Call the points of intersection C and D, and draw a line between them, which cuts AB at some point, as M. Then M is the midpoint of AB, or AB is bisected at M.

(b) With compass show that $AM = MB$.

(c) With compass and protractor show that CD is the perpendicular bisector of AB.

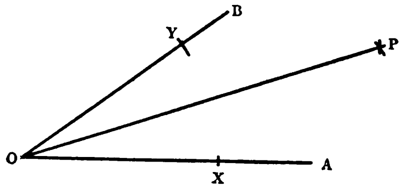
2. *To bisect an angle.*

(a) By folding.

You have already folded a paper twice to make rt. $\measuredangle$s. Fold such a paper again. What is the size of the new angle? Fold again. What is

the size now? Cut out an angle of any size and bisect it by folding.

(b) By using a compass.



Suppose $\angle AOB$ is to be bisected. Put the compass point on O and draw two arcs cutting the sides of the angle at X and Y. Put the compass point on X and then on Y and, without changing the opening, draw two arcs intersecting at P. Draw a line from O through P. What does OP do to $\angle AOB$? Cut out the $\angle AOB$ and show by folding that it is bisected.

CHAPTER FIVE

FURTHER STUDY OF TRIANGLES

A. IMPORTANT LINES IN TRIANGLES

I. In Isosceles Triangles

1. *Draw and cut out a number of isosceles triangles of different sizes and shapes.*

(a) Fold each through the vertex angle so that the two edges lie one on the other.

(b) Do the base angles exactly fit?

(c) Does this test verify the measure with your protractor?

(d) Unfold one or more of these triangles.
Measure the two parts of the base.
In what two ways can you show that the fold bisects the base?

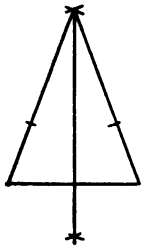
(e) What kind of angles does the fold make as it meets the base?
In two ways prove your answer correct.
What name do you give to a line that makes right angles with another line?

(f) With your protractor measure the two parts of the angle at the vertex.
When the isosceles triangle is folded, do these two angles exactly fit?
What else besides the base of the triangle is bisected by the fold?

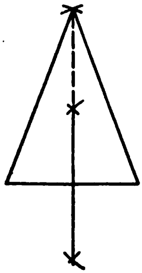
2. *Draw three isosceles triangles exactly equal.*

(a) In $\triangle A$, bisect the vertex $\angle$.
What does the bisector of the vertex $\angle$ of an isosceles $\triangle$ do to the base? How does it meet the base?

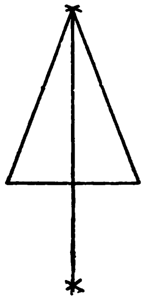
(b) In $\triangle B$, draw the $\perp$ bisector of the base. Extend this $\perp$ bisector. What point of the $\triangle$ does it pass through?



A



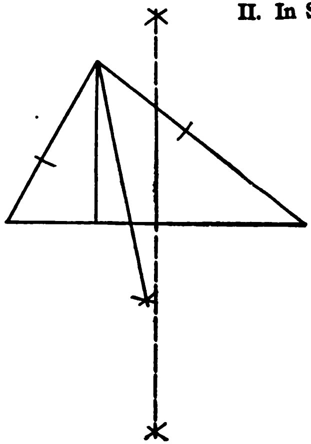
B



C

What does the perpendicular bisector of the base do to the angle at the vertex? Test carefully.

(c) In $\triangle C$, draw a $\perp$ from the vertex to the base. What does this $\perp$ to the base do to the base? What does it do to the $\angle$ at the vertex? Test with compass and protractor.



II. In Scalene Triangles

1. *Draw and cut out three scalene triangles.*

(a) In one of these triangles make three folds.

(1) An altitude

(2) A bisector of the vertex angle

(3) A perpendicular bisector of the base

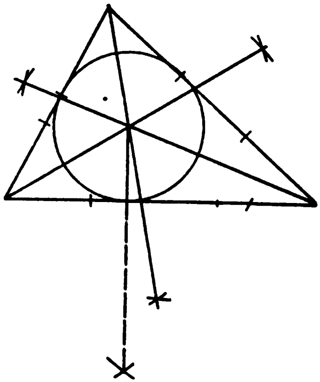
(4) Are these folds the same or are they three separate lines?

(b) In the second triangle, very carefully bisect by folding the three angles of the triangle.

(1) Do these three bisectors meet in a point?

(2) Is it the same point in which the three altitudes meet?

(3) Draw a triangle and accurately construct the bisectors of the angles. Call their intersection O.



From O draw a $\perp$ to the base.

With this $\perp$ as a radius and O as a center draw a circle. Does your circle just touch the three sides of the triangle? If drawn very carefully, it will.

(4) Such a circle is *inscribed* in the triangle, i.e. drawn inside the triangle.

How did you find its center?

How did you find its radius?

(5) Inscribe a circle in an equilateral triangle.

(c) In the third triangle, fold carefully the $\perp$ bisectors of the three sides.

(1) Do these bisectors meet in a point?

(2) Is it the same point as the other intersections?

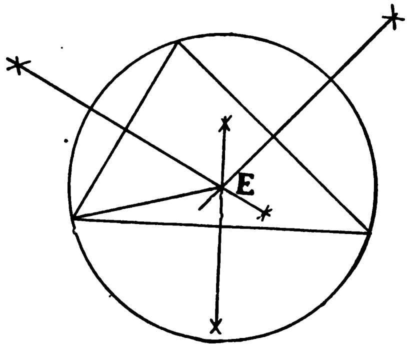
(3) Draw a triangle and accurately construct the $\perp$ bisectors of the three sides.

Call the point of intersection E.

Draw a line from E to one of the vertices of the triangle.

With this line as a radius and E as a center, draw a circle.

Does your circle pass through the other two vertices? If drawn carefully, it will.



(4) Such a circle is called a *circumscribed* circle, i.e., it is drawn about the triangle.

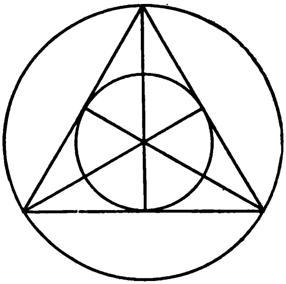
III. Designs

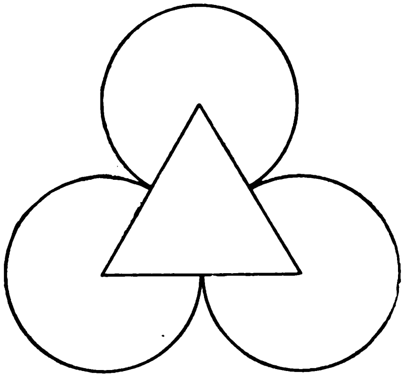
(1) Circumscribe a circle about an equilateral triangle.

(2) Circumscribe a circle about an equilateral triangle, and inscribe a circle in the same triangle.

(3) Draw an equilateral triangle.

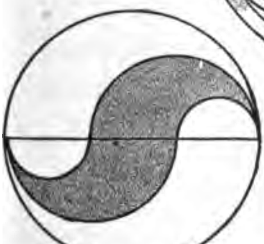
With each vertex as a center and a radius equal to one-half of a side, draw arcs that meet at the midpoints of the sides. Such a figure is called a trefoil and is used extensively in architecture.





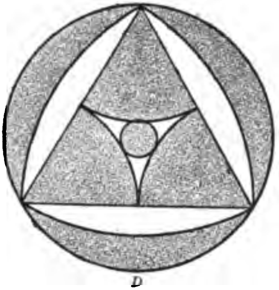
(4) Draw the following patterns or similar ones. Vary them with different shadings or colors. One similar to Fig. B may be made by dividing the diameter in five equal parts instead of three.

A

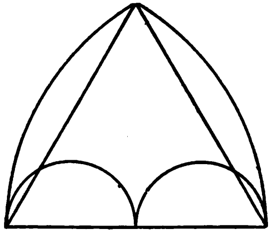


B

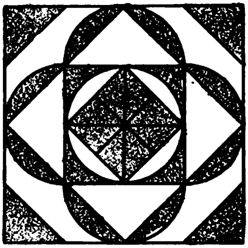
C



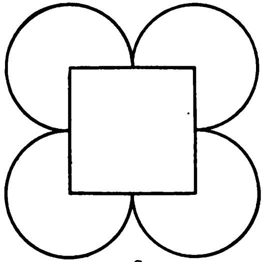
D



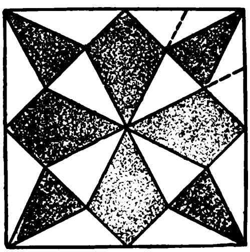
E



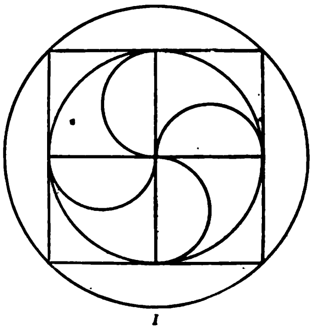
F



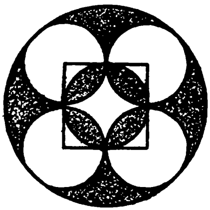
G



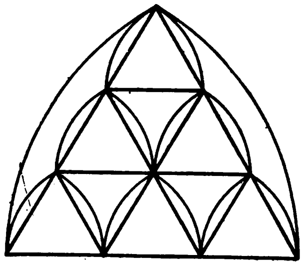
H



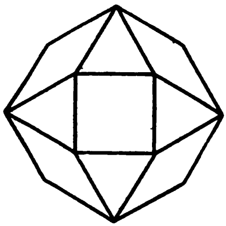
I



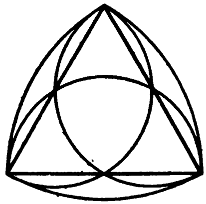
J



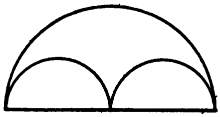
K

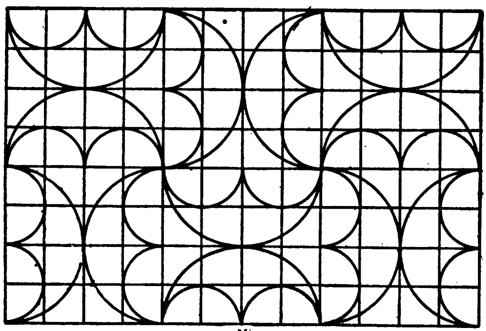


L



M





N

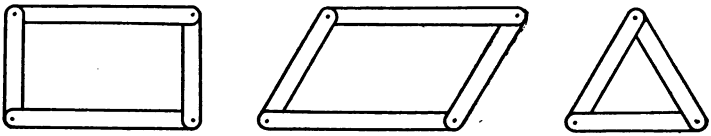
B. THE RIGHT TRIANGLE

I. Its Properties and Practical Uses

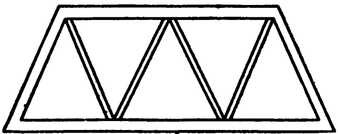
The right triangle is a very interesting figure because of its peculiar properties and its practical uses.

In making trusses for bridges and buildings, the steel beams are put together in the form of triangles for the sake of security, for a triangle cannot change its shape as long as the lengths of its sides do not change.

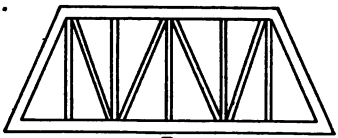
Four rods hinged together may be a rectangle, but pressure or strain may change it to a parallelogram without breaking or changing its sides. Show how this fact is used in telephone brackets on office desks.



Such a change is evidently impossible in a triangle.



A

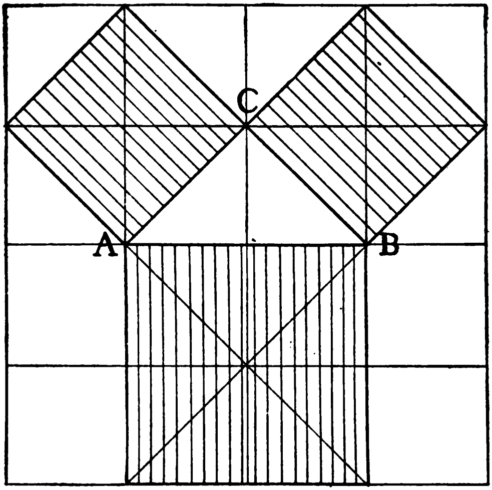


B

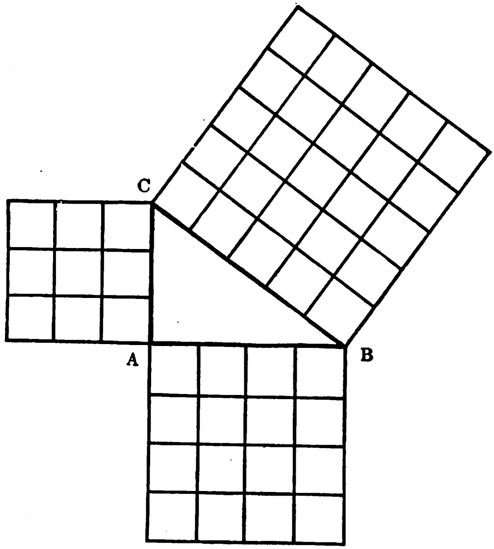
Fig. A shows a kind of truss sometimes used when the strain is not to be very great; but Fig. B shows one of many ways of strengthening it.

If the strain is to be exceedingly great, the second diagonal beam is placed in each rectangle.

The right triangle is a necessary tool for nearly every artisan and craftsman.



1. (a) On a piece of squared paper draw alternate diagonals and shade a square on each side of the right triangle *ABC*. What is the side *AB* called?
   (b) How many small triangles are in the square on the hypotenuse?
   (c) How many are in the square on each leg?
   (d) How does the size of the square on the hypotenuse compare with the sum of the squares on the other two sides?
   (e) The △ *ABC* is a special right triangle. Which kind?
2. (a) Draw a triangle with sides 3, 4, and 5 centimeters long. (If more convenient a half inch may be used for a centimeter.)
   (b) What kind of triangle is formed?
   (c) Carefully construct a square on each side and divide each carefully into square centimeters.



(d) How does the square on the hypotenuse compare with the sum of the squares on the other two sides?

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$$
$$5^2 = 4^2 + 3^2$$
$$25 = 16 + 9$$
$$25 = 25$$

(The line BC squared equals the line AB squared plus the line AC squared. Or, the square on BC equals the square on AB plus the square on AC.)

(e) The truth we have found for two right triangles is proved in geometry for all right triangles. The statement is called a *theorem* and is the most famous of many in geometry. It was discovered by an illustrious Greek named Pythagoras, who lived over 500 years before Christ (569 B.C. to 501 B.C.). It is called the Pythagorean theorem. Over one hundred different

proofs of this theorem have been discovered, one of which was originated by the late President Garfield when he was a boy.

Three thousand years before Pythagoras lived, the Egyptians knew that a triangle whose sides were 3, 4, and 5 units long was a right triangle.

The ancients used this theorem in laying out their temples which had to face a definite direction. They tied twelve knots in a rope of long grass or reeds at equal distances apart, and placed three stakes in such a position that the rope of grass would just reach around them, with one stake at the third knot and one at the seventh.

The men who found the directions in this way were known as "rope-stretchers."

(f) We have seen that a 3-4-5-sided triangle is a right triangle. Would a 6-8-10-sided one be a right triangle also?

(g) Find out how a carpenter or builder uses a 10-foot pole to be sure that the corners of his house are "square," i.e., right angular.

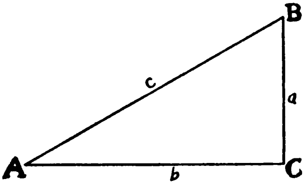
(h) A statement that two numbers or quantities are equal is called an *equation*.

Formulas are usually written as equations; as,

$$S = l \cdot w$$
$$12 = 4 \cdot 3$$

(i) From the Pythagorean theorem we have the equation

$$c^2 = a^2 + b^2$$



if c is the hypotenuse, and a and b are the legs of a right triangle.

(j) We know that $\frac{1}{4} = .25$.
Extract the square root of each side of the equation.

$$\frac{1}{2} = .5$$

Are the square roots equal?
If we extract the square root of both sides of an equation or formula, we still have an equation.

(k) In the rt. triangle, $c^2 = a^2 + b^2$

Therefore $c = \sqrt{a^2 + b^2}$

If $a = 3$
and $b = 4$,

$$\begin{aligned} c &= \sqrt{3^2 + 4^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

3. Find the hypotenuse of the right triangles whose perpendicular sides are as follows:

	a	b	c
(a)	5″	12″	
(b)	45′	60′	
(c)	1.5″	2″	
(d)	15′	36′	
(e)	8″	15″	
(f)	2′ 6″	6′	
(g)	4′	7′ 6″	
(h)	8″	10″	
(i)	7″	24″	
(j)	2″	3″	
(k)	3′ 6″	12′	
(l)	10′	12′	

(m) Find the length of the diagonal of a square whose side is 5″; 7″; $2\frac{1}{2}$″.

(n) A telegraph pole is 30′ high. How long a guy wire will be needed to fasten it to a stake 16′ from the foot of the pole, if one foot is allowed at each end for fastening?

(o) *Suggestions.*

(1) In problem 3 (b) you were to find the hypotenuse of a right triangle whose sides were 45 and 60.

Did you recognize 45 as 3×15?
and 60 as 4×15?

Therefore the hypotenuse is 5×15 or 75.

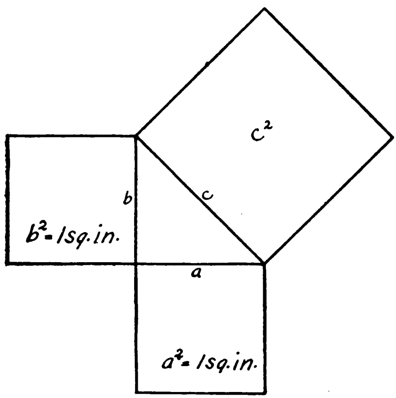
(2) In 3 (a) you found another group, 5, 12, and 13, that makes a right triangle.

(3) How many problems in the set are based on others already found?

(4) Only six of these twelve problems need to be worked out in full. The others may be solved by inspection.

Is it not worth while to try to factor the numbers first?

4. (a) Draw an isosceles right triangle with legs 1 inch long, and draw the squares on the three sides.



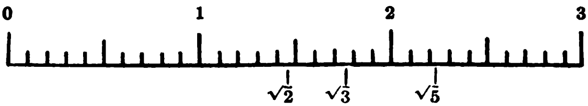
(b) According to the Pythagorean theorem how does the square on c compare with a^2? With b^2? How many square inches are in c^2?

(c) Measure the length of c with your ruler. Can you measure it exactly?

If $\qquad c^2 = 2$

then $\qquad c = \sqrt{2}$

(d) On a straight line carefully marked off in inches and tenths of inches, lay off the exact length of the hypotenuse c.



We see that the mark comes between 1.4 and 1.5 inches. If we could divide our tenths of an inch into tiny hundredths of an inch, the mark would come between 1.41 and 1.42 inches.

(e) Find the square root of 2 correct to four decimal places.

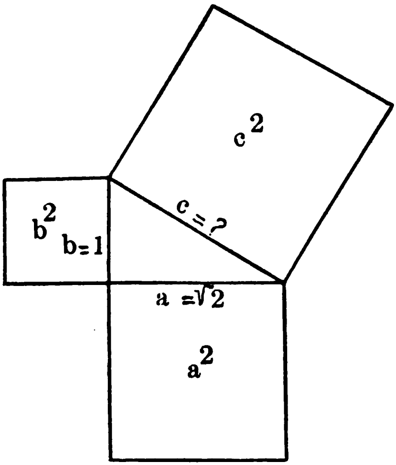
$$\sqrt{2} = 1.4+ \qquad (1.4+)^2 = 1.96+$$
$$\sqrt{2} = 1.41+ \qquad (1.41+)^2 = 1.9881+$$
$$\sqrt{2} = 1.414+ \qquad (1.414+)^2 = 1.999396+$$
$$\sqrt{2} = 1.4142+ \qquad (1.4142+)^2 = 1.999961+$$

We see that $\sqrt{2}$ is a real and exact measure of the hypotenuse, while the measures on the scale line are only approximate, but growing more nearly correct as we add decimal places.

5. (a) Draw a right triangle with a base equal to c or $\sqrt{2}$ and an altitude of 1 inch.

$$a^2 = ?$$
$$b^2 = ?$$
$$c^2 = ?$$
$$c = ?$$

(b) On the scale lay off the length of the hypotenuse c. Between what two marks on the scale does it lie?



(c) Find the square root of 3 correct to four decimal places, and find the squares of the successive values.

6. (a) Draw a right triangle whose legs are 1 and 2 inches, respectively. What is the area of the square on the hypotenuse?

(b) What number represents the exact length of the hypotenuse? Find its approximate value on the scale.

7. (a) What are the lengths of the sides of a right triangle whose hypotenuse is exactly $\sqrt{6}$?

(b) Such numbers as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, etc., whose values can be represented exactly by lines, but only approximately by ordinary numbers, are sometimes called *irrational numbers*.

$\sqrt{3}$ is just as real as 1.73205, and is more exact.

Many times it is more convenient to use the irrational form instead of the ordinary form.

(c) How would you draw a square whose area is exactly 3 square inches? 4 sq. in.? 5 sq. in.? 6 sq. in.? 10 sq. in.?

CHAPTER SIX

PARALLEL LINES

A. MEANING OF PARALLEL LINES

1. What does *parallel* mean?

Explain what is meant by parallel lines.

2. Is it possible that two lines could never meet and yet *not* be parallel?

3. Draw a diagonal in the upper face of a cube. Can such a diagonal be prolonged far enough to meet any lower edge?

Are they parallel?

4. How must the position of two parallel lines be limited in your definition?

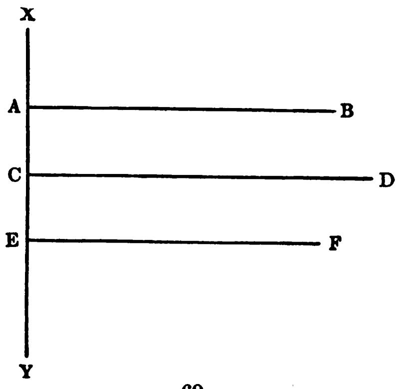
B. CONSTRUCTION OF PARALLEL LINES

I. With Protractor

1. *To draw parallels with protractor.*

Draw three or more lines ⊥ to XY.

What kind of lines have you drawn?

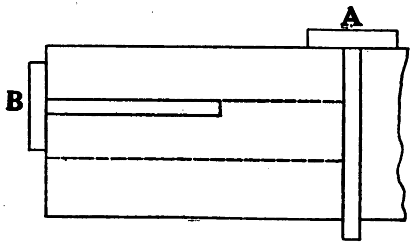


II. With T Square

1. *To draw parallels with T square.*

Draftsmen and carpenters draw parallel lines thus with T squares.

Suppose a carpenter wishes to cut off 4-inch strips from a 12-inch board. First he "squares off" the end by placing the T square in position *A*. Then across the true end he notes the 4-inch and 8-inch marks, places his square as in position *B* and draws his lines for sawing.

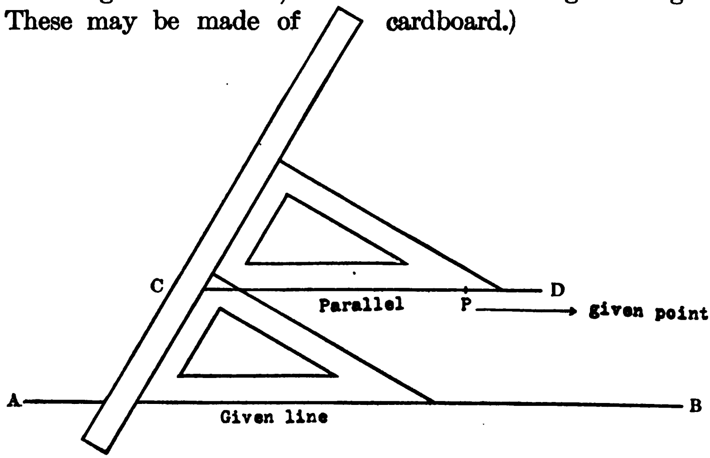


III. With Drawing Triangle

1. *To draw parallels with drawing triangle.*

For finer work, draftsmen use a right triangle, usually made of celluloid.

(Pupils should have two right triangles, one with the acute angles 30° and 60°, the other an isosceles right triangle. These may be made of cardboard.)



(a) AB is the given line and P a point *not* on AB, through which a line is to be drawn $\parallel$ to AB.

(b) Place the drawing triangle with any edge, preferably the hypotenuse, exactly fitting the given line. Place your ruler so as to fit one of the other edges of the drawing triangle.

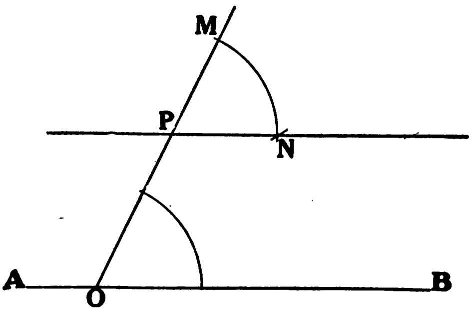
(c) Hold the ruler still and slide the triangle along it, keeping the edge carefully fitted against the ruler, until the hypotenuse passes through P.

(d) Draw a line CD along the hypotenuse passing through P. Then CD is $\parallel$ to AB.

NOTE: With ruler and drawing $\triangle$ using the legs of the $\triangle$, i.e. the $\perp$ sides, show how one can draw a $\perp$ to a given line from a given point.

IV. With Ruler and Compass

1. *To draw parallels with ruler and compass.*



Let P be the point through which a line is to be drawn $\parallel$ to AB.

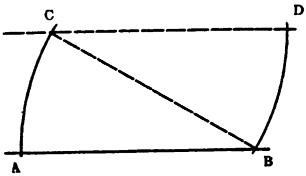
(a) Through P draw any line meeting AB at O.

(b) At P, draw an $\angle\ MPN = \angle\ POB$.

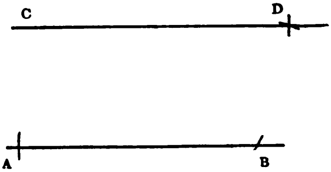
(c) The line PN is $\parallel$ to AB.

2. *Second method with ruler and compass.*

This method is given in a French book published in 1723.



A



B

Let C be the point through which a line is to be drawn $\parallel$ to AB.

(a) Draw any line from C to AB, as BC.

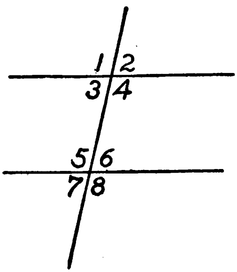
(b) With BC as a radius and B and C as centers, draw the $\frown$s AC and BD.

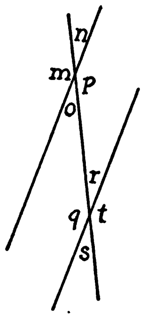
(c) Measure the $\frown AC$ and make $\frown BD = \frown AC$.

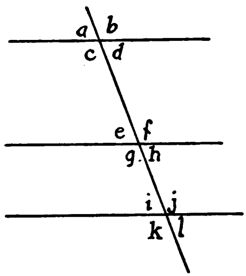
(d) The line CD is $\parallel$ to AB.

(e) Fig. B shows the same construction with only parts of the lines drawn.

C. PARALLELS CUT BY A TRANSVERSAL







Draw several sets of two or more parallels, with a line drawn across or cutting the parallels. Such a line is called a *transversal* (meaning *turned across*).

If two parallels are cut by a transversal, eight angles are formed. For convenience we give these angles different names, depending upon their position.

I. Angles Made by a Transversal

1. The angles lying *between* the parallels are called *interior* angles (inside the parallels), as angles 3, 4, 5, and 6.

2. The angles lying *without* the parallels are called *exterior* angles (outside the parallels), as angles 1, 2, 7, and 8.

3. (a) The angles lying on *opposite* or *alternate* sides of the transversal are called *alternate* angles, as angles 3 and 6 or angles 1 and 8.

(b) If angles 3 and 6 are a pair of alternate interior angles, what may you call angles 1 and 8?

(c) Select and measure the alternate interior angles. How do they compare?

4. The angles lying on the *same* side of the transversal and on the *same* or *corresponding* sides of the parallels are called *corresponding* angles, as angles 2 and 6, or angles 3 and 7.

5. The fourth method of constructing two parallel lines was by making two equal corresponding angles.

6. Select and measure all the corresponding angles in these figures.

What do you conclude about their equality?

II. Vertical Angles

1. The angles lying opposite each other at the same vertex are called *vertical* angles, as angles 1 and 4, or angles 2 and 3.

2. If any two straight lines intersect, how many pairs of vertical angles are formed?

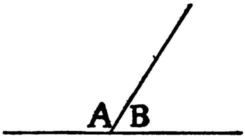
3. Select and measure the vertical angles in the given figures.

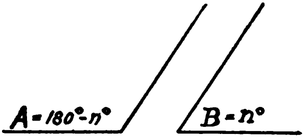
III. Supplementary Angles

1. Any two angles whose sum is 180°, or a straight angle, are *supplementary angles.*

Each angle is the supplement of the other.

2. When one straight line meets another straight line a pair of supplementary angles is formed.

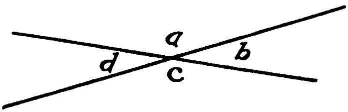


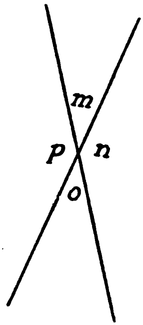


3. How many degrees are in two supplementary angles? How many right angles does their sum make? How many straight angles?

4. If $\angle A$ contains 120°, how many degrees are in its supplement, $\angle B$?

5. If two straight lines intersect, four pairs of supplementary angles are formed.





Find the pairs of supplementary angles in each of these figures.

6. (a) With your protractor draw the following angles.

(b) Compute and draw their supplements.

(1)	(2)	(3)	(4)
120°	$22\frac{1}{2}$°	117°	$\frac{8}{9}$ rt. $\angle$
45°	15°	29°	$1\frac{1}{2}$ rt. $\angle$
30°	115°	81°	$\frac{5}{6}$ rt. $\angle$
90°	65°	122°	$1\frac{3}{5}$ rt. $\angle$
150°	185°	$\frac{2}{3}$ rt. $\angle$	1 rt. $\angle$

7. (a) In the figure of the parallels, page 72, what is the sum of $\angle 2 + \angle 4$?

(b) Since $\angle 2 = \angle 6$, we may substitute $\angle 6$ for its equal, $\angle 2$, in the equation, $\angle 2 + \angle 4 = 2$ rt. $\measuredangle$s.

$$\therefore \angle 6 + \angle 4 = 2 \text{ rt. } \measuredangle s.$$

(c) What kind of angles are $\measuredangle$s 6 and 4?

8. Complete the following statements correctly by filling the blanks with either the word "equal" or "supplementary."

(a) If two straight lines intersect, the vertical angles are ———.

(b) If two parallel lines are cut by a transversal,

(1) the alternate interior angles are ———.

(2) the alternate exterior angles are ———.

(3) the corresponding angles are ———.

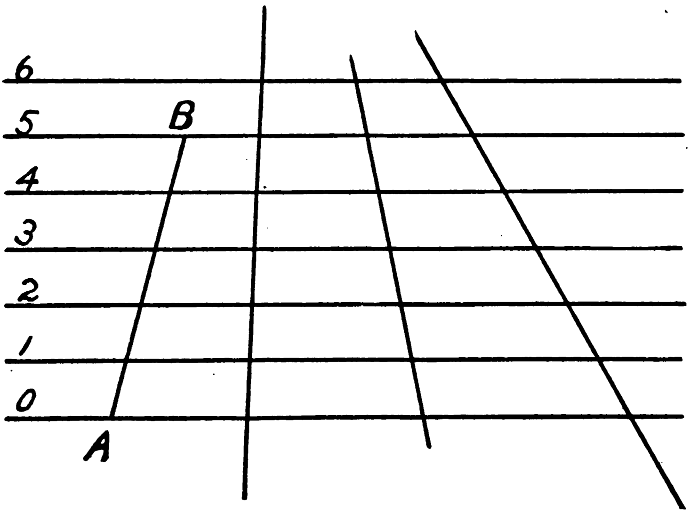
(4) the two interior angles on the same side of the transversal are ———.

(5) the exterior angles on the same side of the transversal are ———.

(c) In the figure of the parallels, page 72, find all the other angles when $\angle 2 = 60°$; when $\angle 5 = 110°$; when $\angle 3 = 45°$.

D. PRACTICAL USE OF PARALLELS

1. *To divide a line into equal parts.*



(a) Draw a series of seven ||s that are equal distances apart, or use a sheet of paper ruled at equal intervals.

(b) Across the series draw several transversals not || to each other.

(c) With your compass measure the different parts into which each line is divided. How do the parts of the same line compare?

(d) If one wishes to divide a line 5″ or 10″ long into five equal parts, it is easy enough, for it can be done with a ruler or tape measure. But it is not so easily done if the line is 4″ and a fraction. By using parallels it may be done accurately.

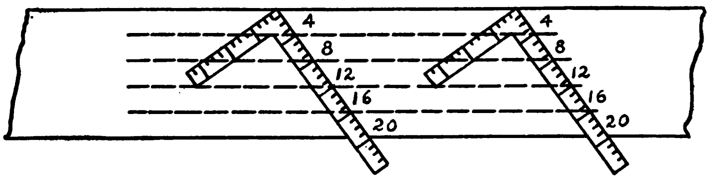
(1) Number the ||s 0, 1, 2, 3 . . . .

(2) Place one end of the line on 0 and the other end on the 5th ||, as the line *AB*.

(3) Mark where each of the other parallels crosses the line.

(4) These marks divide it into five equal parts.

2. On page 70 we saw how a carpenter could divide a 12-inch board into three strips. Suppose he wishes to divide it into five strips of equal width.



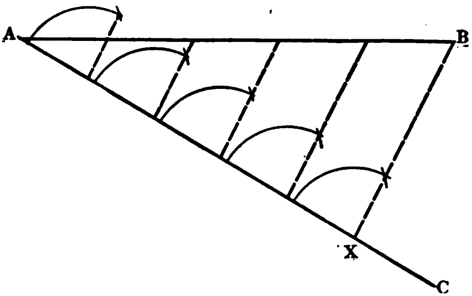
He places his square across the board in such a position that he can mark off five equal parts, as at 4, 8, 12, 16, and 20. (By placing it at another angle he might use 3, 6, 9, 12, and 15.)

He moves the square to another position and repeats the process. Through the corresponding marks he draws straight lines as guides for sawing.

(a) Show that this process is correct.

(b) Could he use marks 4, 8, 12, 16, and 20 for one set and 3, 6, 9, 12, and 15 for the other? Explain clearly. Could he use marks 2, 4, 6, 8, and 10?

3. *To divide a line into any number of equal parts with compass and ruler.*



To divide AB into five equal parts.

(a) Draw AC making an $\angle$ with AB.

(b) On AC mark off five equal parts of any convenient length. From X, the fifth point, draw BX.

(c) Through each point on AX draw a line $\parallel$ to BX.

(d) These $\parallel$s divide AB into 5 equal parts. Explain.

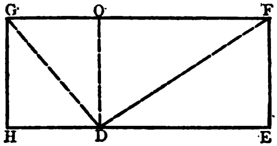
4. (a) On squared paper draw any scalene triangle ABC. Draw an altitude CD, bisect it at O, and through the point of bisection draw a line GF parallel to the base.

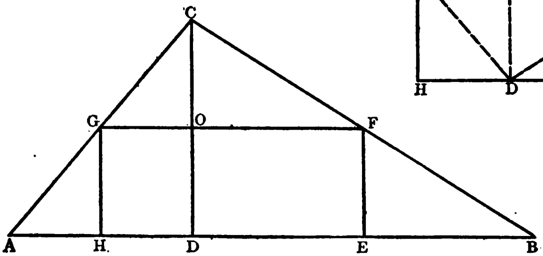
(With protractor draw $GF \perp$ to CD at O.)

Draw GH and FE parallel to CD.

Cut out the $\triangle$ so drawn and by folding carefully along GF, FE, and GH, show the

$\square\ EFGH = \frac{1}{2} \triangle\ ABC$.





Show that $CF = BF$

$CG = AG$

$AH = HD$

$DE = EB$

$\triangle\, AHG = \triangle\, GOC$

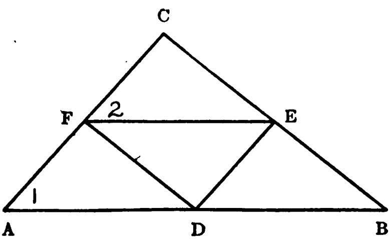
(What part of $\square$ *GHDO* does each $\triangle$ make?)

Show that $\triangle\, COF = \triangle\, FEB$.

(What part of $\square$ *DEFO* is $\triangle$ *EBF?* Is $\triangle$ *OFC?* What part of $\triangle$ *ABC* is in the $\square$ *HEFG?*)

(*b*) If a line is drawn parallel to one side of a triangle, bisecting another side, what does it do to the third side?

5. Draw any scalene triangle. Bisect each side and join the midpoints.



Cut out the $\triangle$ and cut off one of the smaller $\triangle$s. Try to fit it on the other $\triangle$s. How do these lines joining the midpoints divide $\triangle$ *ABC?*

Compare $\angle$ 1 with $\angle$ 2. What is the relative position of *AB* and *FE?* What other lines are parallel?

CHAPTER SEVEN

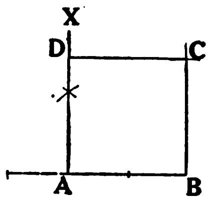
QUADRILATERALS

A. CONSTRUCTION OF QUADRILATERALS

At first we drew squares and rectangles on squared paper. Now we shall learn to draw them with compass and ruler.

1. *To draw a square.*

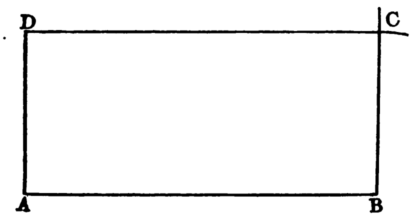
(a) Suppose AB is the side of the square. At A draw $AX \perp$ to AB. With compass measure $AD = AB$. With B and D as centers and an opening $= AB$, make two arcs intersecting at C. Draw BC and CD. Then $ABCD$ is a square.



(b) Test and see that it has all the necessary features of a square.

2. *To draw a rectangle.*

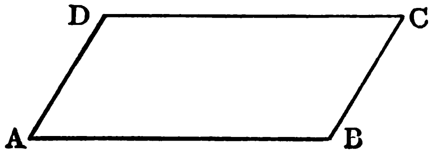
(a) On the sides of a rt. $\angle$ lay off AB the length and



AD the width of the ▭. With D as a center and an opening of the compass $= AB$, draw an ⌒ above B. With B as a center and an opening $= AD$, draw an arc intersecting the first one at C. Draw BC and CD.

(b) Show that $ABCD$ is a rectangle.

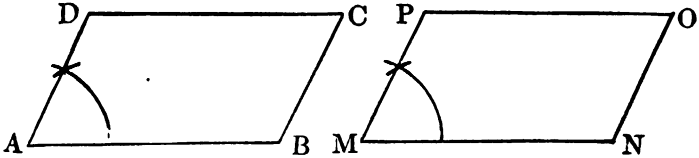
3. *To draw a parallelogram.*



(a) If AB and AD are the two sides of the ▱ let them meet at any ∠.

(b) How can you find point C and complete the ▱?

4. *To draw a parallelogram equal to a given parallelogram.*



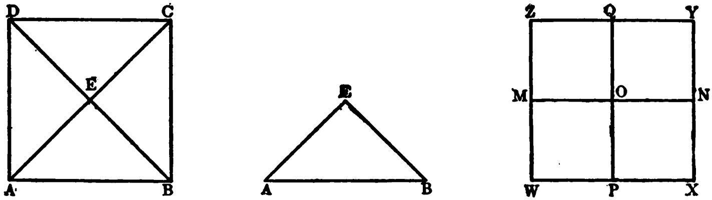
Let $ABCD$ be the given ▱.
Take $MN = AB$.
At M make an ∠ = ∠ BAD, and take $MP = AD$.
How can you complete the ▱?

B. THE SQUARE

I. Relation of Lines and Angles

1. Draw and cut out a square.

2. Fold carefully on two diagonals.

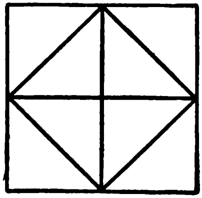


3. What does each diagonal do to the ∠s at each corner or vertex? How large is each ∠?

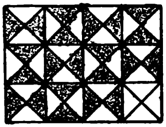
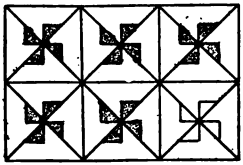
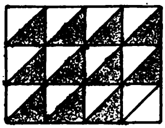
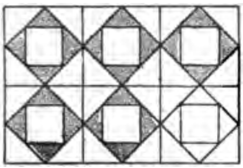
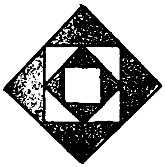
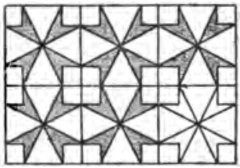
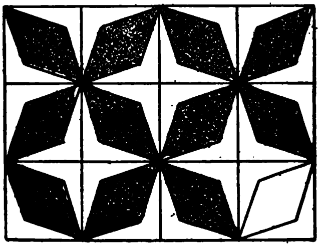
What is the size of ∠ E?

4. How does each diagonal cut the other?
5. Are they perpendicular to each other?
6. How many ◬ are formed?
7. How do they compare with each other?
8. Bisect each side of the square and fold across.
9. How does the $\triangle$ *ABE* compare with the $\square$ *PXNO*?

10. *MN* and *PQ*, the lines joining the midpoints of the sides of a square, are its *medians* or *midjoins*. Sometimes these lines are called the *diameters*.



11. (a) Join the ends of the medians of a square in succession.
 (b) What kind of figure is formed?
 (c) What part of the original square does it contain?

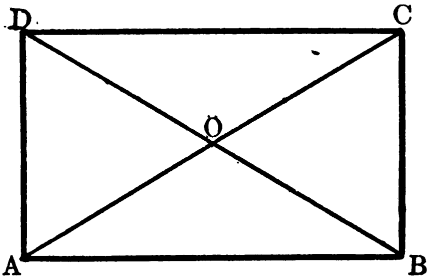
12. By drawing medians and diagonals of a group of squares and by shading some of the small triangles or other parts, very pretty designs may be made. These are used principally in linoleum and tile floors.

(a) Copy the given designs on squared paper or make original ones.

C. THE RECTANGLE

I. Relation of Lines and Angles

1. Draw and cut out any rectangle. Fold on both diagonals. How does each diagonal divide the ▭? (See page 36.)



2. Cut out ⚠ *AOB* and *BOC* and fit them on ⚠ *COD* and *AOD* respectively.

3. Are the diagonals equal? Are they ⊥ to each other?

4. Do they bisect each other?

5. Do the diagonals bisect the angles at the vertices or corners? In what kind of rectangle is this done?

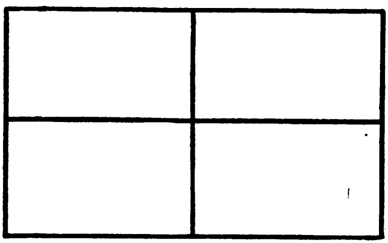
6. What kind of △ is *AOB?* △ *BOC?* △ *COD?* △*AOD?*

7. Find all possible pairs of equal angles in the figure.

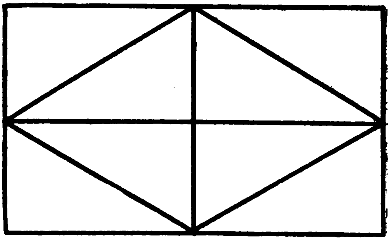
8. (a) Connect the midpoints of the opposite sides of a rectangle.

(b) What is the name of these lines?

(c) How do they divide the rectangle?



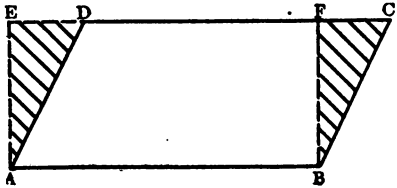
9. (a) Join the ends of the medians in succession.
(b) What kind of figure is formed?
(c) What part of the original rectangle does it contain?



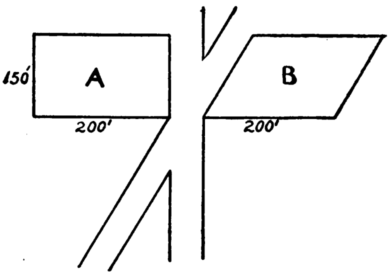
D. THE PARALLELOGRAM

I. Area

1. (a) Draw a ▱ $ABCD$. Draw AE and $BF \perp$ to AB. What is the shape of $ABFE$?



(b) Any $\perp$ between the bases of a ▱ is its *altitude*.
(c) Cut off the $\triangle BFC$ and fit it on $\triangle ADE$.
(d) How does the size of the ▱ compare with that of the ▭? Compare their bases; their heights.
(e) What is the formula for the area of the rectangle? Of the parallelogram?
(f) If $AB = 15$ and $AE = 8$, what is the area of ▭ $ABFE$? Of ▱ $ABCD$?



(g) Draw several ▱s and show that their areas are equal to the ▭s with same dimensions.

2. (a) Two roads intersecting at an $\angle$ of 35° divide a

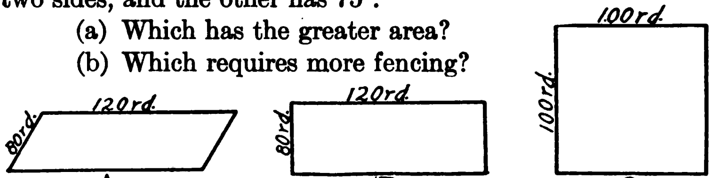
man's land into two tracts as in the figure. Find the area of each part *A* and *B*.

(b) Will each part require the same amount of fencing?

3. Two fields in the shape of parallelograms have equal bases and equal altitudes. One has an angle of 50° between two sides, and the other has 75°.

(a) Which has the greater area?

(b) Which requires more fencing?



A B C

4. Figures A, B, and C represent three fields, each of whose perimeters measures 400 rods. It is said that pioneers traded land with the Indians on a basis of perimeters. An Indian had plot *C* and was offered his choice of *A* and *B*. Which offer should he take, if either?

Explain answer.

5. (a) A boy has his choice of three lots for use as a garden. Each of these takes 20 rods of fencing.

The first lot is in the shape of a square; the second is a rectangle; the third, a rhomboid.

Which shall he choose in order to have the largest garden?

(b) How large is the garden measured in square rods? in square feet?

Is its width longer or shorter than the width of an ordinary city lot?

(c) Draw to any convenient scale the plan of the garden, allowing 15 inches all around it inside the fence, and allotting $\frac{1}{4}$ to tomatoes, $\frac{1}{8}$ to cabbage, and $\frac{1}{8}$ to small garden truck, as radishes, lettuce, etc., and the remainder to potatoes.

(d) How many tomato plants must be set out if the rows are to be 18 inches apart and the plants in each row are 15 inches apart?

(e) How many seed potatoes must the gardener buy?

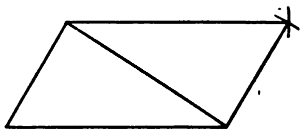
(f) If you have had a garden of your own, you may calculate a fair yield from such a garden.

(g) How much larger is the chosen garden than the rectangular one, if its length is four times its width?

II. Relation of Lines and Angles

1. (a) Draw any parallelogram.

(b) Cut out the figure and fold on one diagonal. Do the two triangles fit?



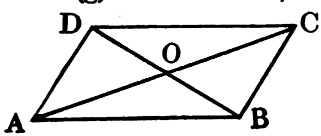
(c) Cut along the diagonal. Now can they be made to fit?

(d) How does the diagonal of a ▱ divide the figure?

(e) How do the opposite sides of a ▱ compare in length? in position?

(f) How do the ∠s at the opposite corners compare in size?

(g) If $\angle A = 60°$, find the size of ∠s B, C, and D.



(h) Draw a ▱ and its two diagonals.

(i) Are the two diagonals equal? Are they ⊥ to each other?

(j) Do they bisect each other?

(k) Do they bisect the ∠s at the vertices?

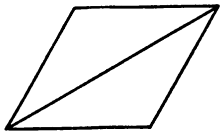
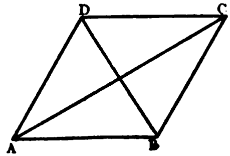
(l) Do they make any pairs of equal triangles?

(m) Is any one of these △s isosceles?

(n) Find all possible pairs of equal angles.

III. The Rhombus

1. (a) Draw an equilateral parallelogram.
   What is the name of this $\square$?

   (b) Fold on the diagonal.
   Do the two triangles fit?
   (c) Draw the other diagonal.
   (d) Are the two diagonals equal?
   (e) Are they perpendicular to each other?
   (f) Do they bisect each other?
   (g) Do they bisect the $\angle$s at the vertices?
   (h) Do they make any pairs of equal $\triangle$s? Name them.
   (i) Are these $\triangle$s isosceles?
   (j) What kind of $\triangle$ is ABC? BCD?
   (k) If $\angle\ BAD$ is 60°, find the size of all other $\angle$s in the figure (15 more in all).

E. SUMMARY

In which of these quadrilaterals

1. are the diagonals equal?
2. do the diagonals bisect each other?
3. are the diagonals perpendicular to each other?
4. do the diagonals bisect the angles at the vertices?
5. do the diagonals form pairs of equal $\triangle$s?
6. are these pairs of $\triangle$s isosceles?

CHAPTER EIGHT

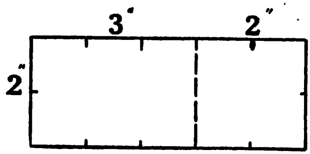
MEASURED AND UNMEASURED LINES

A. PROBLEMS FROM RECTANGLES

I. Parentheses

1. Draw a rectangle $3'' \times 2''$. Find its area.
(If more convenient, draw to scale $\frac{1}{2}''$ to $1''$.)

2. (a) Add two inches to the length of this same rectangle. What is the area of the added part?



$$\begin{aligned} S \text{ of } \square &= 1 \times w \\ &= (3 + 2) \times 2 \\ &= 6 + 4 \\ &= 10 \text{ sq. in.} \end{aligned}$$

(b) Such an expression as $(3 + 2) \times 2$ means that the sum of 3 and 2 is to be multiplied by 2. It may be written thus: $2\,(3 + 2)$ without the multiplication sign and is read "two times the quantity three plus two."

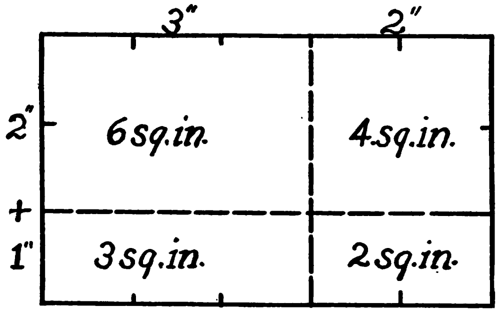
(c) Its value may be found by adding the 3 and 2 and then multiplying the sum 5 by 2, which gives 10; or each part of the quantity may be multiplied and the partial products added.

$$\begin{array}{r} 3 + 2 \\ \times \quad\; 2 \\ \hline 6 + 4 = 10 \end{array} \qquad \Bigg| \qquad \begin{array}{r} 3 + 2 = 5 \\ \times \qquad\quad 2 \\ \hline 10 \end{array}$$

3. What is the perimeter of the $\square$?

$$\begin{aligned} P_{\square} &= 2\,(l + w) \\ &= 2\,(3 + 2 + 2) \\ &= 6 + 4 + 4 \\ &= 14 \text{ inches} \end{aligned} \qquad \text{or} \qquad \begin{aligned} P_{\square} &= 2l + 2w \\ &= 2\,(3 + 2) + 2 \cdot 2 \\ &= 10 + 4 \\ &= 14 \text{ inches} \end{aligned}$$

4. (a) Find the area of a rectangle $5'' \times 3''$.
   (b) Express its dimensions if the length is increased by $1''$; by $2''$; by $3''$.
   (c) Find the area of each.
   (d) Find the perimeter of each.
5. (a) Estimate the dimensions of a page in your book. Estimate its area.
   (b) Then measure carefully and compute the area from the exact measurements.
   (c) Suppose the page were $2''$ longer; how large would it be?
6. (a) Draw a rectangle $3'' \times 2''$. Increase the length by $2''$ and the width by $1''$.



$$\begin{aligned} S_{\square} &= l \cdot w \\ &= (3 + 2) \times (2 + 1) \\ &= 5 \times 3 \\ &= 15 \text{ sq. in.} \end{aligned}$$

   (b) Into how many small $\boxed{s}$ is the figure divided?
   (c) What is the area of each small $\square$?
   (d) What is the sum of these areas?
   (e) The expression $(3 + 2) \times (2 + 1)$ means that the sum of 3 and 2 is to be multiplied by the sum of 2 and 1. It may be written without the

multiplication sign, as $(3+2)(2+1)$, and it is read, "the quantity 3 plus 2 multiplied by the quantity 2 plus 1."

(f) The multiplication may be made in two ways.

$$\begin{array}{l} 3+2 \\ \underline{2+1} \\ 6+4 \\ \quad\ \underline{3+2} \\ 6+7+2=15 \end{array} \quad \text{or} \quad \begin{array}{r} 3+2=5 \\ 2+1=\underline{3} \\ 15 \end{array}$$

The first form shows the areas of the small rectangles.

7. What is the perimeter of the rectangle given in example 6?

$$\begin{aligned} P_{\square} &= 2\,(l+w) \\ &= 2\,(3+2+2+1) \\ &= 6+4+4+2 \\ &= 16 \text{ inches} \end{aligned} \quad \text{or} \quad \begin{aligned} P_{\square} &= 2\,l+2\,w \\ &= 2\,(3+2)+2\,(2+1) \\ &= 2\,(5)+2\,(3) \\ &= 10+6 \\ &= 16 \text{ inches} \end{aligned}$$

8. (a) Find the area of a $\square$ 7″ by 4″.
   (b) Express its dimensions if the length is increased by 3″ and the width by 2″.
   (c) Find its area.
   (d) Find its perimeter.

9. (a) Suppose your book were 2″ longer and 1″ wider, then estimate its area and perimeter.
   (b) Measure carefully and compute both area and perimeter.

10. (a) Estimate the dimensions and area of your school room.
   (b) Measure and compute its area and perimeter.
   (c) Suppose it were 5 ft. longer and 3 ft. wider; what would be its area and perimeter?

11. Find how much varnish would be required to put two coats of varnish on the floor and wainscoting. (If your room has no wainscoting, assume one $2\frac{1}{2}$ ft. high.)

12. (a) Measure the space necessary for each school desk and for aisles.

(b) Would it require more or less space to have the desks face at right angles to their present position, and how much?

(c) Could more or fewer seats be placed leaving the same width of aisles between the rows and around the room?

13. (a) Measure five other rectangular objects in the school.

(b) Estimate their areas and perimeters.

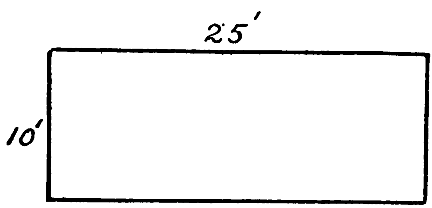
(c) Then compute with their dimensions slightly increased.

II. Ratio

1. (a) A flower bed is $25' \times 10'$. How many times longer is it than wide?

$$\frac{l}{w} = \frac{25}{10} = \frac{5}{2} \text{ or } 2\tfrac{1}{2}$$

(b) This statement means that the length 25 has the same relation to the width 10 as 5 has to 2. In other words, the length is $2\frac{1}{2}$ times the width.



The statement $\frac{l}{w} = \frac{5}{2}$ may be read in two ways:

"the ratio of the length to the width is the same as the ratio of 5 to 2," or "l is to w as 5 is to 2."

(c) If we asked what is the relation between the width and the length, we would turn the fraction upside down.

$$\frac{w}{l} = \frac{10}{25} = \frac{2}{5}$$

This means that the width is $\frac{2}{5}$ of the length.

(d) The comparison or relation between two numbers expressed as a fraction is called a *ratio*.

(*Ratio* is the Latin word for relation.)

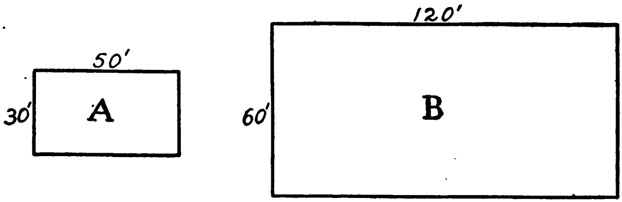
(e) To find the ratio between two quantities, measure them in the same way, i.e. by the same unit of measure, and write these measures as a fraction reduced to lowest terms.

(f) What is the ratio between the length of your desk and that of your teacher's?

(g) Measure your desk. Perhaps it is 24″ long. Measure your teacher's desk. It may be $3\frac{1}{2}$ ft. long. You cannot compare these unless both are measured by the same unit. Measured in inches their ratio is $\frac{24}{42} = \frac{4}{7}$. Measured in feet, their ratio is $\dfrac{2}{3\frac{1}{2}} = \dfrac{4}{7}$.

2. (a) Two gardens are $30' \times 50'$ and $60' \times 120'$ respectively. How many times longer is the fence surrounding the second than that of the first?

(b) If the first fence cost $8.00, what will the second cost?



Solution of (a).

$P \text{ of } A = 2(l + w)$	$P \text{ of } B = 2(l + w)$
$= 2(50 + 30)$	$= 2(120 + 60)$
$= 2(80)$	$= 2(180)$
$= 160$	$= 360$

$$\frac{P_B}{P_A} = \frac{360}{160} = \frac{9}{4}$$

Solution of (b).

To find the cost of the second fence we must find the number of dollars that will have the same ratio to 8 that 9 has to 4.

Let n = the number of dollars in the cost of the second fence.

Then $\frac{n}{8}$ = ratio of these two costs.

But this ratio must equal the ratio of the two perimeters, which is $\frac{9}{4}$. Therefore, $\frac{n}{8} = \frac{9}{4}$

Multiply both sides of this equation by 8.

$$\cancel{8}\left(\frac{n}{\cancel{8}}\right) = \overset{2}{\cancel{8}} \times \frac{9}{\cancel{4}}$$

If equals are multiplied by equals the products must be equal.

$$n = 18$$

Therefore the cost of the second fence is $18.

Proof: Does $\frac{18}{8} = \frac{9}{4}$?

3. Note: (a) $\frac{n}{8} = \frac{9}{4}$ is an equation.

(b) *To solve an equation* is to find the value of the letter which stands for the unknown number.

(c) To solve this equation we chose to multiply both sides by 8. Why? We wanted to get rid of the denominators 8 and 4. By multiplying by the lowest common denominator, we got an equation without fractions.

(d) An equation is like a balance. Therefore we may add the same number to both sides, subtract the same number from both sides, or multiply or divide both sides by the same number without destroying the balance.

4. A room is 18 feet long. The ratio of the width to the length is $\frac{3}{4}$. Find the width of the room.

Solution. Let w = number of feet in the width of the room.

$$\text{Then } \frac{w}{18} = \frac{3}{4}$$

$$\overset{2}{\cancel{36}}\left(\frac{w}{\cancel{18}}\right) = \overset{9}{\cancel{36}} \times \frac{3}{\cancel{4}} \qquad \times \text{ by 36. Why?}$$

$$2w = 27$$

$$w = 13\tfrac{1}{2} \qquad \div \text{ by 2.}$$

Therefore, the width = $13\frac{1}{2}$ ft.

Proof: Does $\dfrac{13\frac{1}{2}}{18} = \dfrac{3}{4}$?

5. The dimensions of two rectangles have the same ratio. The first rectangle A is $9' \times 15'$. The second, B is $25'$ long. How wide is it?

$$\text{In } \square\, A, \frac{w}{l} = \frac{9}{15}$$

$$\text{Therefore, in } \square\, B, \frac{w}{25} = \frac{9}{15}$$

$$\overset{3}{\cancel{75}}\left(\frac{w}{\cancel{25}}\right) = \overset{5}{\cancel{75}} \times \frac{9}{\cancel{15}} \qquad \times \text{ by 75.}$$

$$3w = 45$$

$$w = 15 \qquad \div \text{ by 3.}$$

Therefore, the width of $\square$ B = 15 ft.

Does $\frac{15}{25} = \frac{9}{15}$?

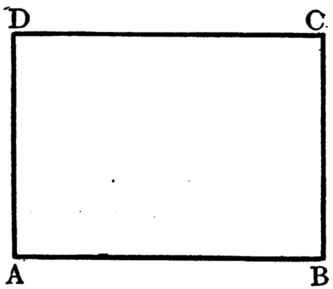
6. A room is $15' \times 18'$. The rug on the floor is 12 ft. wide. How long must it be to have the same ratio between its length and width as between the dimensions of the room?

7. On a map a distance of 150 miles is represented by a line $1\frac{1}{2}''$ long. Two cities are $5''$ apart on the map. What is the distance between them?

8. In a geography find a map and read the scale. Measure as accurately as possible the lines between four pairs of large cities. Calculate their distances.

9. Draw an angle of 45°. What is the ratio between it and its complement? Between it and its supplement?

10. $\angle A$ is 30°. Find the ratio between $\angle A$ and its complement. Between $\angle A$ and its supplement. Between its complement and its supplement.



11. Measure the length and width of the $\square$ $ABCD$ to the nearest tenth of an inch. What is the ratio of the length to the width?

Find its perimeter.

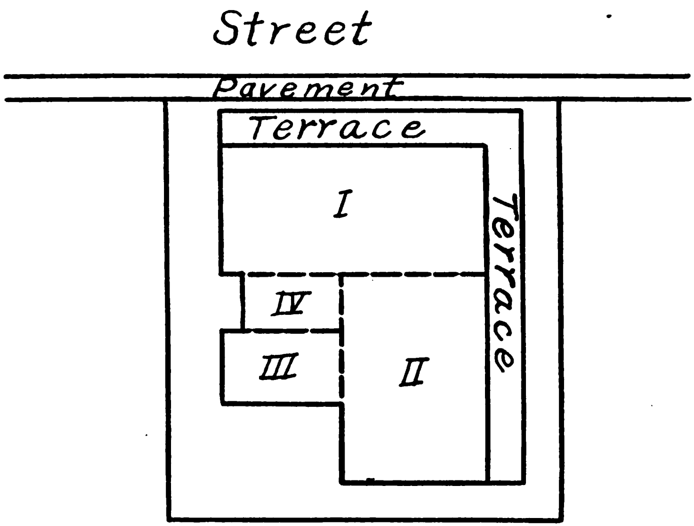
12. Measure your school room correct to the nearest half-foot. Draw a plan of it on the scale of $5'$ to $1''$. Locate the teacher's desk and the rows of pupils' desks.

13. Draw a diagram of your war garden on a scale of $10'$ to $1''$. (Use $20'$ to $1''$, if more convenient.)

14. Draw a plan of the first floor of your house on a scale of $10'$ to $1''$.

15. This diagram of a large club house is drawn on a scale of $100'$ to $1''$. All angles are right angles.

(a) What is the width of the street? Of the pavement?



(b) What are the dimensions of the plot?

(c) Room I is a reception hall.
Room II is a dining hall.
Room III is the kitchen.
Room IV is the serving room.
What are the dimensions of each?

(d) What is the area of each?

(e) What is the area of the terrace?

(f) What part of the plot is covered by the building including the terrace?

(g) What is the ratio of the width to the length of the plot?

16. In some newspaper or magazine, find some floor plans of houses. Some will give the dimensions. Others will give the scale of the drawing.

From the dimensions find the scale, and vice versa.

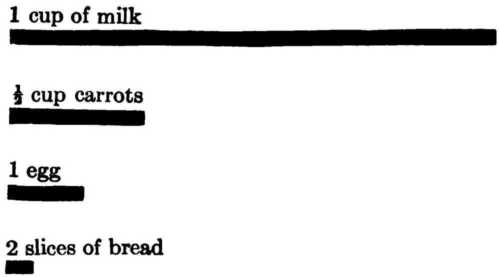
III. Graphs of Ratios

1. (a) Ratios are used practically every day by most people, but in newspapers and magazines pictures are made of them, because they are more effective than the numbers. These pictures are called *graphs*. *Graph* comes from the Greek word which means *to write* or *draw*.

(b) The following paragraph and graph is taken from one of the U. S. Food Administration bulletins.

Milk is the chief food for lime. It is much richer in it than other common foods. These lines stand for lime, the top one for the lime in a cup of milk, the others for the lime in a serving of some other foods. Notice how much more there is in milk than in the others.

Amount of Lime in



Milk is the cheapest food for lime. Buy milk. You and your children need its lime.

How much more effective is that long line for milk and the short lines for egg and bread, than for one to be told that the amount of lime in a cup of milk is almost $6\frac{1}{2}$ times that in an egg?

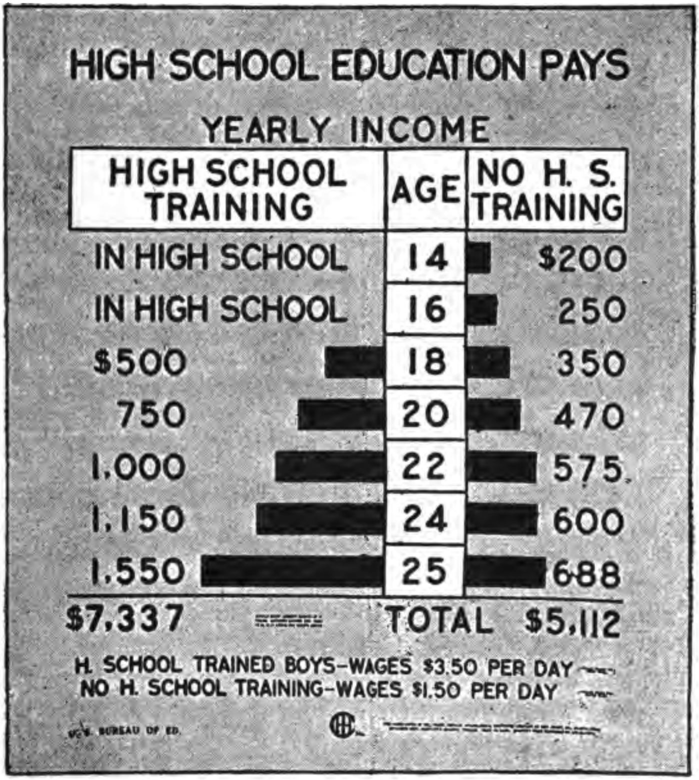
(c) Measure each of these lines to the nearest tenth of a centimeter and find the ratio of the lime in milk to that in the carrots, egg, and bread.

2. The following graph was taken from a school paper.

(a) If this graph is correctly drawn, the bar represented by \$250 should be $\frac{1}{2}$ of the \$500-bar.

(b) Measure carefully to see if that ratio holds.
(c) Test also for \$750 and \$1000.
(d) Compare the \$200 and \$600 bars.

Does a High School Education Pay?



How much is your head worth? It is worth as much as you put in it. How much is your body worth? One dollar and fifty cents a day. That is, from your head down you are worth less than two dollars a day.

The above statistics are taken from data gathered by the Bureau of Education of the United States Government.

At the age of fourteen, a boy in high school is not usually earning anything. Often, by working after school, he earns enough to buy his clothing and books. A boy not in school can earn about two hundred dollars a year, which must pay all of his expenses.

A boy in school at the age of sixteen is making himself more efficient. A boy sixteen years old not in school earns only two hundred fifty dollars a year.

The average boy graduates from high school at the age of eighteen. Then he is able to earn at least five hundred dollars, while the boy without a high school education is now earning only about three hundred fifty dollars a year.

By the time a boy with a high school education has reached the age of twenty, statistics prove that he is able to earn seven hundred fifty dollars; while the boy with whom we are comparing him earns about four hundred seventy dollars.

The salaries of each are gradually increased until they are each twenty-five years old. By that time the high school graduate receives a salary of about one thousand five hundred fifty dollars a year, while the man with no high school education earns a salary of about six hundred eighty-eight dollars a year.

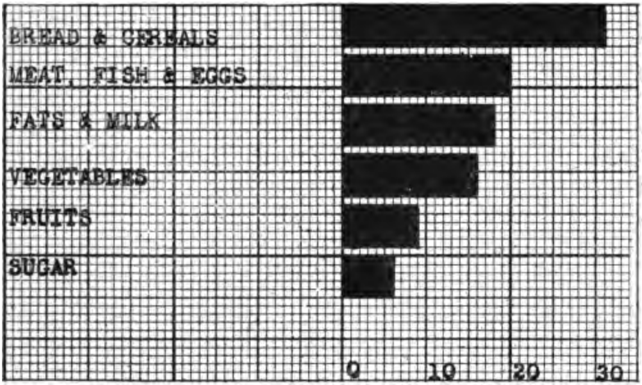
Within the thirteen years from the time they were fourteen until they were twenty-five, inclusive, the high school student and graduate has earned seven thousand three hundred thirty dollars. The man working for thirteen years and having no high school education has earned five thousand one hundred twelve dollars.

Does a high school education pay?

3. Percentage as a ratio.

When we say that 31% of a person's diet is grains or cereals we mean that the ratio of the quantity of cereals that he eats to the total quantity of his food is equal to the ratio of 31 to 100.

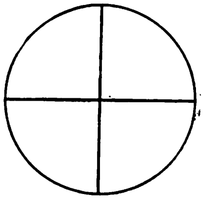
From the following graph read the per cent that each item of food is of his diet.



4. (a) The pictures of the ratios are sometimes shown in a circle graph instead of in bar graphs.

(b) Around the center of a circle there are four right angles or 360°.

(c) Each right angle is $\frac{1}{4}$ of 360° and 25% is $\frac{1}{4}$ of 100%. Therefore each right angle represents 25%.



(d) If the amount of cereal in Ex. 3 is 31%, how large an angle will represent the cereal?

Let n = the number of degrees in the angle.

$$\text{Then } \frac{n}{360} = \frac{31}{100}$$

$$\overset{10}{\cancel{3600}} \times \frac{n}{\cancel{360}} = \frac{31}{\cancel{100}} \times \overset{36}{\cancel{3600}} \qquad \times 3600$$

$$10\,n = 1116$$

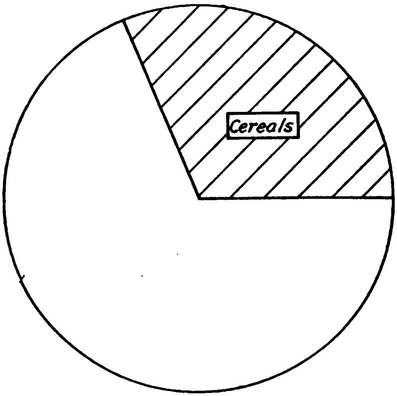
$$n = 111.6 \qquad \div\ 10$$

Therefore the $\angle$ for cereal $= 111.6°$

$= 111°\ 36'$

(e) Draw a large circle. At the center make an angle as nearly 111° 36′ as can be made with your protractor. Shade this angle in some way and mark it *cereals*.

(f) Adjacent to the angle make other angles representing the correct per cents for the other



articles of food. Shade each angle in a different way, either with lines running in different directions, or with different kinds of lines or with cross bars.

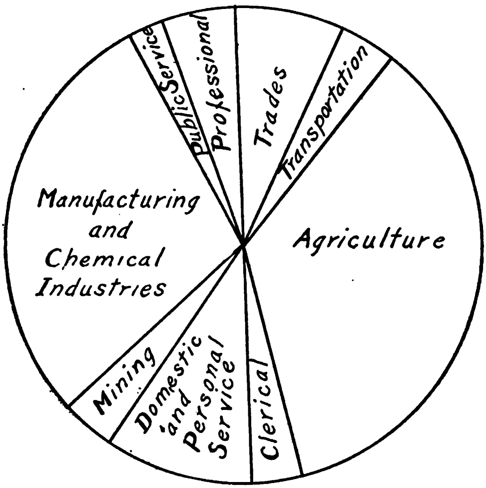
SAMPLES OF SHADING

5. In your geography and science books you will find many graphs. Find several and explain their meaning.

6. (a) The following circle graph shows the approximate distribution of the people of the United States according to their fields of work.

(b) By measuring the angles find the per cent of population in each occupation.

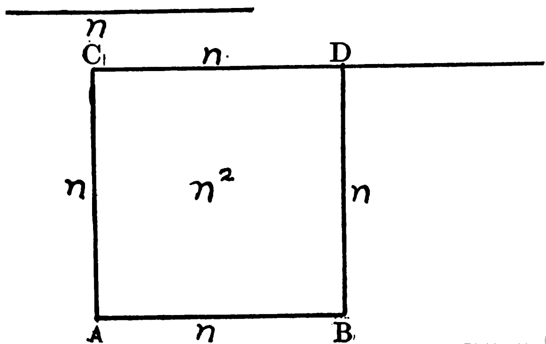
(c) The sizes of four angles with fractional number of degrees are given. Others can be measured accurately. The ∡ for transportation and mining are each 10.8°. The ∡ for clerical and public service are each 7.2°.



B. PROBLEMS WITH UNMEASURED LINES

I. Perimeters and Areas

1. (a) Draw a line of any unmeasured length. Draw a square using this line as a side.



(b) Each pupil will probably draw a line different in length from the others, but each line will have a certain number of inches or parts of inches in its length, although we do not know just what the number is. In place of the number we do not know, suppose we use the first letter of the word *number* and say the side of the square is n inches long, or more briefly,

$$AB = n \text{ inches.}$$
$$AD = n \text{ inches.}$$

(c) Now if one happens to draw his line 2″, then the area of his square is 2^2 or 4 square inches. But no matter how long the lines are, the area of each square is $n \cdot n$, or n^2, and the perimeter of each is $4n$. We see then that

$$(1) \qquad n \cdot n = n^2$$
$$\text{and } (2) \quad n + n + n + n = 4n$$

What do you call the figure 2 used in n^2 in the first statement? What does it show? What do you call the figure 4 used in $4n$ in the second statement? What does it show?

(d) We see that n^2 stands for area and its picture is a square; but $4n$ stands for length and its picture is a line.

$$n \quad + \quad n \quad + \quad n \quad + \quad n$$

Perimeter = $4n$. (Scale $\frac{1}{2}$)

(e) Measure your own line and find your own values of n^2 and $4n$.

(f) We might have used any other letter instead of n, as m, p, x, y, z, or any other.

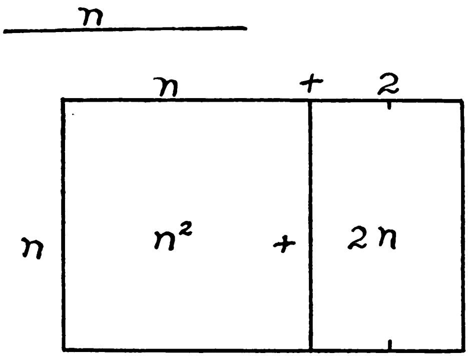
2. If the picture of n^2 is a square whose side is n, can you draw a picture of $4n^2$? Remember that $(2n)^2 = 2n \cdot 2n = 2^2 \cdot n^2 = 4n^2$.

3. Draw a picture of $\frac{1}{2}n$, $2n$, $\frac{3}{4}n$, $1\frac{1}{2}n$, $9n^2$.

4. What line can you draw in the last picture to make one of $4\frac{1}{2}n^2$?

5. Use a different length of line (any that is not too large) for each different letter, and then draw pictures of $2m$, p^2, $m+p$, x^2, $y+z$, $3x$, $2x+3y$, $m+2n+3p$, $\frac{1}{2}x+\frac{1}{3}y$.

6. (a) Draw n^2. Add 2 cm. to the length.



(b) What is the shape of the new figure?

(c) What is the area of each part?

(1) We saw in the preceding exercise that $2n$ meant $n+n$ and then its picture was a line. In this figure we see that $2n$ may mean $2 \times n$ and then its picture is a rectangle.

(d) Find the area and perimeter of the new figure.

$$
\begin{aligned}
S_{\square} &= l \times w \\
&= (n+2)\,n \\
&= n^2 + 2n
\end{aligned}
$$

$$
\begin{aligned}
P_{\square} &= 2\,(l+w) \\
&= 2\,(n+2+n) \\
&= 2\,(2n+2) \\
&= 4n+4
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
P_{\square} &= 2l + 2w \\
&= 2\,(n+2) + 2n \\
&= 2n + 4 + 2n \\
&= 4n+4
\end{aligned}
$$

(e) Measure n in centimeters and find the area and perimeter.

(f) What is the unit of measure of $n^2 + 2n$? Of $4n + 4$?

7. (a) Draw any line x cm. long.

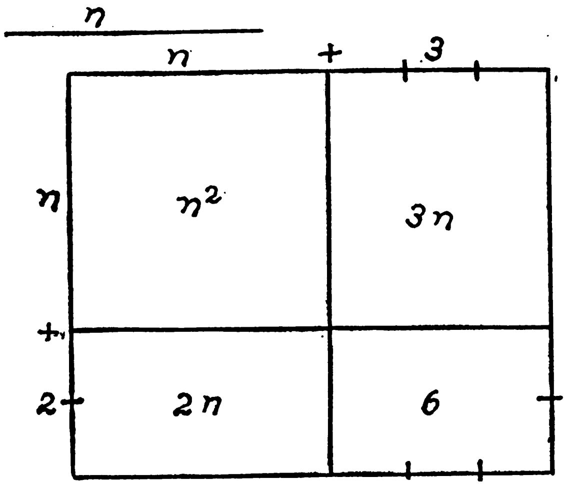
(b) Draw a rectangle $x + 3$ centimeters by x centimeters.

(c) Compute its area and perimeter.

(d) Measure x to the nearest tenth of a centimeter and find the ratio of its width to its length.

(e) Find the numerical values of the area and perimeter.

8. (a) Draw n^2. Increase its length 3 centimeters and its width 2 centimeters.



(b) What is the area of each of the four small rectangles? What is their sum?

(c) The sum expressed as $n^2 + 3n + 2n + 6$ is said to have four terms. n^2 is one term; $3n$ is another; and so are $2n$ and 6 other terms.

(1) An expression having only one term is called a *monomial.* Literally the word means "one named."

(2) An expression having more than one term is called a *polynomial.* Literally it is "many named."

(3) A polynomial having only two terms is called a *binomial;* and one having only three terms is called a *trinomial.*

(4) Derivations:

$$\left.\begin{array}{ll}\text{Mono} & \text{— one}\\ \text{Bi} & \text{— two}\\ \text{Tri} & \text{— three}\\ \text{Poly} & \text{— many}\end{array}\right\} + \text{nomen} = \text{name}\left\{\begin{array}{l}\text{Monomial}\\ \text{Binomial}\\ \text{Trinomial}\\ \text{Polynomial}\end{array}\right.$$

You are familiar with the prefixes *mono-* and *bi-* in the words *monoplane* and *biplane* or *bicycle*, and with the prefix *tri-* in the word *tricycle* or *trisect.*

The word *nomen* or *name* is found in our word *nominate*, which means *to name for an office.*

(d) We find that the sum of the four rectangles gives a polynomial of four terms. Can you combine two of the terms so as to make a trinomial out of it?

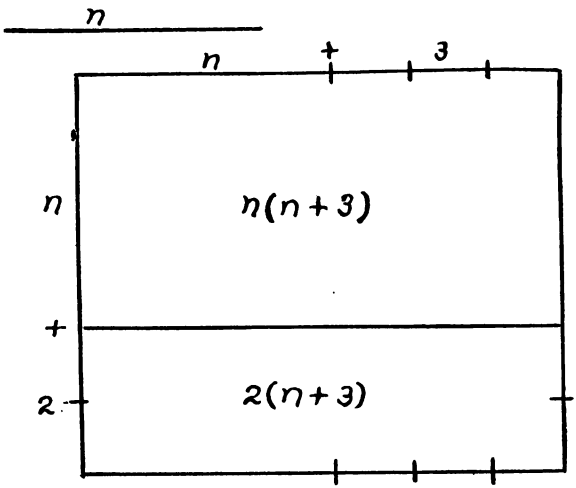
(1)
$$\begin{aligned} S_{\square} &= l \cdot w \\ &= (n+3)(n+2) \\ &= n^2 + 3\,n + 2\,n + 6 \\ &= n^2 + 5\,n + 6 \end{aligned}$$

(2) Process by multiplication.

$$\begin{array}{l} n + 3 \\ \underline{n + 2} \\ n^2 + 3\,n \\ \underline{ + 2\,n + 6} \\ n^2 + 5\,n + 6 \end{array}$$

(3) Find the area or S as the sum of two small rectangles.

$$\begin{aligned} S_{\square} &= n\,(n + 3) + 2\,(n + 3) \\ &= n^2 + 3\,n + 2\,n + 6 \\ &= n^2 + 5\,n + 6 \end{aligned}$$



(e) Find the perimeter.

$$\begin{aligned} P_{\square} &= 2\,(l + w) \\ &= 2\,(n + 3 + n + 2) \\ &= 2\,(2\,n + 5) \\ &= 4\,n + 10 \end{aligned}$$

or

$$\begin{aligned} P &= 2\,l + 2\,w \\ &= 2\,(n + 3) + 2\,(n + 2) \\ &= 2\,n + 6 + 2\,n + 4 \\ &= 4\,n + 10 \end{aligned}$$

What is the name of the unit of measure of the perimeter?

(f) Measure n in centimeters and find the value of the area and perimeter. Find the value of each term separately before adding.

(g) (1) Draw another unmeasured line m inches long.

(2) Draw a square on the line.

(3) Increase its length by 3 inches and its width by 1 inch.

(4) Find the area of the square; of the rectangle.

(5) Find the perimeter of the square; of the rectangle.

(6) Measure your line m to the nearest tenth of an inch and find the numerical values of the areas of the square and of the rectangle.

(7) Find the numerical values of the perimeters of both figures.

(h) (1) Draw squares on four other unmeasured lines.

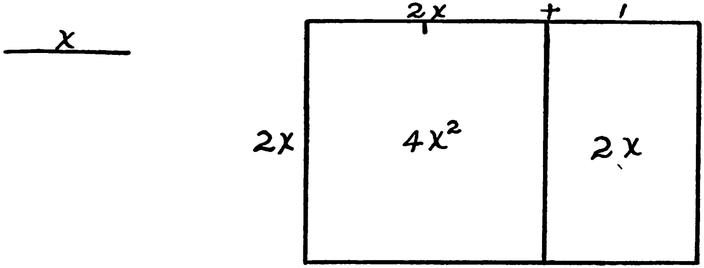
(2) Increase the length and width of each by different numbers of inches.

(3) Find the area and perimeter of each figure.

(4) Find the numerical measure of each area and perimeter.

9. (a) Draw an unmeasured line x inches long.

(b) Draw a square $2x$ inches on a side. What is its area?



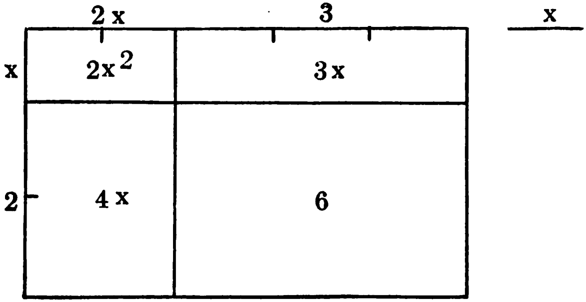
(c) Add 1 inch to the length of the square. Find the area and perimeter of the new figure.

(d) Measure the line x and find the value of the expressions for the area and the perimeter. To find the value of $4x^2$, square the measure of x before multiplying by 4.

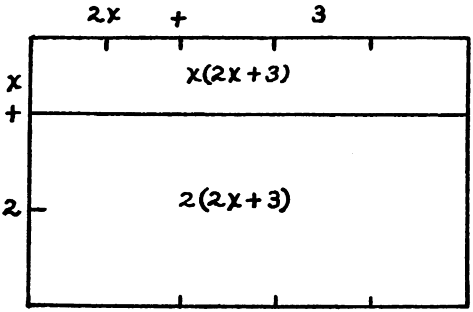
(e) Find the ratio of the width to the length.

(f) Find the ratio of the area of the original square to the area of the rectangle.

10. (a) Draw a rectangle $2x + 3$ inches long and $x + 2$ inches wide.



(b) What is the area of each of the four small rectangles? Express the sum as a trinomial.



(c) Omit one of the dividing lines and indicate the area as the sum of two rectangles.

(1) *Solution of* (b).

$$S_{\square} = l \times w$$
$$= (2x + 3)(x + 2)$$
$$= 2x^2 + 7x + 6$$

Multiplication

$$\begin{array}{r} 2x + 3 \\ x + 2 \\ \hline 2x^2 + 3x \\ + 4x + 6 \\ \hline 2x^2 + 7x + 6 \end{array}$$

(2) *Solution of* (c).

$$\begin{aligned} S &= x(2x+3) + 2(2x+3) \\ &= 2x^2 + 3x + 4x + 6 \\ &= 2x^2 + 7x + 6 \end{aligned}$$

11. Find the perimeter of the rectangle given in example 10.

12. (a) Measure the line x and find the numerical number of square inches in the area.

(b) Find the number of inches in the perimeter.

(1) *Solution of* (a).

$$\begin{aligned} l &= 2x + 3 \\ w &= x + 2 \end{aligned}$$

By measurement, $x = \frac{3}{4}$ in.

$x^2 = (\frac{3}{4})^2 = \frac{9}{16}$ sq. in.

We found $S = 2x^2 + 7x + 6$

$= 2(\frac{9}{16}) + 7(\frac{3}{4}) + 6$ Subst. $\frac{3}{4}$ for x

$= \frac{9}{8} + \frac{21}{4} + 6$

$= 1\frac{1}{8} + 5\frac{1}{4} + 6$

$= 12\frac{3}{8}$ sq. in. area of ▭

Note: Prove this area is correct by multiplying the length in inches by the width in inches.

(2) *Solution of* (b).

In example 11 we found $P = 6x + 10$.

By measurement, $x = \frac{3}{4}$

$P = 6x + 10$

Therefore ($\therefore$) $P = 6(\frac{3}{4}) + 10$

$= 4\frac{1}{2} + 10$

$= 14\frac{1}{2}$ inches.

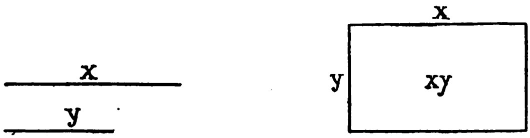
Note: The word *therefore* is used a great many times in mathematical problems. The symbol for it is three small dots put in the form of an equilateral triangle ($\therefore$).

13. Find the areas and perimeters of rectangles whose dimensions follow.

In each case, measure the given line and check the results.

	Length	Width	S	P
(a)	$2n+3$	$n+4$	....	....
(b)	$a+5$	$a+3$	....	....
(c)	$3y+4$	$2y+3$	....	....
(d)	$2x+5$	$x+1$	....	....
(e)	$2x+5$	$2x+3$	....	....
(f)	$a+4$	$a+2$	....	....
(g)	$b+7$	$b+1$	....	....
(h)	$2c+10$	$c+3$	....	....
(i)	$n+6$	$n+6$	....	....
(j)	$5a+3$	$4a+1$	....	....
(k)	$5a+2$	$5a+2$	....	....
(l)	$2n+5$	$2n+5$	......	....

14. (a) Draw two unmeasured lines, x and y centimeters long, and draw a rectangle of these lines.



(b) Find area.

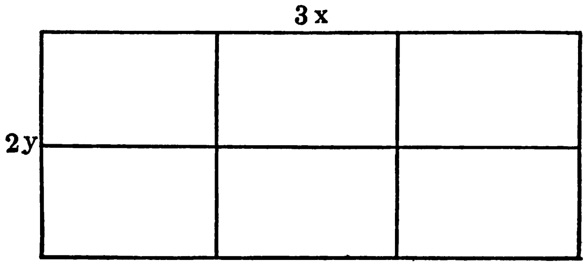
$$
\begin{aligned}
S_{\square} &= l \times w \\
&= x \times y \\
&= xy
\end{aligned}
$$

Just as $2x$ means 2 times x, so xy means x times y. The times sign is omitted between two factors when one or both of them are letters. In arithmetic 2×3 could not be written 23, for digits in arithmetical numbers have place value and the 2 means 2 tens or 20.

(c) Find perimeter.

$P = 2(l + w)$	or	$P = 2l + 2w$
$= 2(x + y)$		$= 2x + 2y$

15. (a) Draw a rectangle $3x$ centimeters long and $2y$ centimeters wide.



(b) Find the area.

$$S_{\square} = lw$$
$$= 3x \cdot 2y$$
$$= 6xy$$

How many rectangles the size of xy does it contain?

(c) Find the perimeter.

$P = 2(l + w)$	or	$P = 2l + 2w$
$= 2(3x + 2y)$		$= 2(3x) + 2(2y)$
$= 6x + 4y$		$= 6x + 4y$

(d) Measure x and y in centimeters; find the value of S and P and the ratio of the dimensions.

16. Find the areas and perimeters of rectangles having the following dimensions:

	Length	Width		Length	Width
(a)	$2x$	$3y$	(e)	$\frac{2}{3}c$	$\frac{3}{4}d$
(b)	$5a$	$6b$	(f)	y	$\frac{1}{2}z$
(c)	$4m$	$3n$	(g)	$2\frac{1}{2}a$	$4b$
(d)	$\frac{1}{2}x$	$\frac{1}{3}y$	(h)	$\frac{3}{4}a$	$\frac{8}{15}x$

17. (a) In finding the product of two factors that have unlike letters, what do you do with the *coefficients?*

(b) What do you do with the two letter factors?

(c) How do you write the product of two like letter factors?

18. (a) How does the picture of xy differ from that of x^2?

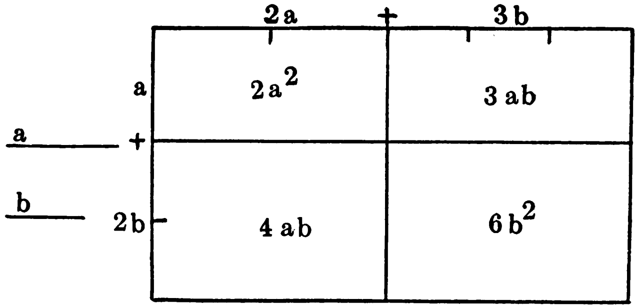
(b) What is the picture of $x + y$?

(c) Can the picture of x^2 ever be a line?

(d) Can the picture of xy ever be a line?

19. (a) Draw two unmeasured lines, a and b centimeters long, respectively.

(b) Draw a rectangle whose length is $2a + 3b$ centimeters and whose width is $a + 2b$ centimeters.



6 cm. long. (Scale ½)

(c) What are the dimensions of each of the four small rectangles?

(d) What is the area of each?

(e) What is the sum of these parts of the large rectangle?

(f) Can you give the sum as a trinomial?

(g) Find the area by multiplication.

$$S_{\square} = lw$$
$$= (2a + 3b)(a + 2b)$$
$$= 2a^2 + 7ab + 6b^2$$

$$\begin{array}{r} 2a + 3b \\ a + 2b \\ \hline 2a^2 + 3ab \qquad \\ + 4ab + 6b^2 \\ \hline 2a^2 + 7ab + 6b^2 \end{array}$$

(h) Find the perimeter.

$$\begin{aligned} P &= 2\,(l + w) \\ &= 2\,(2\,a + 3\,b + a + 2\,b) \\ &= 2\,(3\,a + 5\,b) \\ &= 6\,a + 10\,b \end{aligned}$$

or

$$\begin{aligned} P &= 2\,l + 2\,w \\ &= 2\,(2\,a + 3\,b) + 2\,(a + 2\,b) \\ &= 4\,a + 6\,b + 2\,a + 4\,b \\ &= 6\,a + 10\,b \end{aligned}$$

(i) Evaluation.

(1) Area.

By measurement, $a = 3$ cm.
$b = 2$ cm.

From these measurements what is the value of a^2? of ab? of b^2?

$$\begin{aligned} S_{\square} &= 2\,a^2 + 7\,ab + 6\,b^2 \\ &= 2\,(3^2) + 7(3)(2) + 6\,(2)^2 \\ &= 2\,(9) + 7\,(6) + 6\,(4) \\ &= 18 + 42 + 24 \\ &= 84 \text{ sq. cm.} \end{aligned}$$

or

$$\begin{aligned} l &= 2\,a + 3\,\mathrm{b} \\ &= 2\,(3) + 3\,(2) \\ &= 6 + 6 \\ &= 12 \\ w &= a + 2\,b \\ &= 3 + 2\,(2) \\ &= 3 + 4 \\ &= 7 \\ S &= lw \\ &= 12 \cdot 7 \\ &= 84 \text{ sq. cm.} \end{aligned}$$

(2) Perimeter.

$$\begin{aligned} P &= 2\,l + 2\,w \\ &= 6\,a + 10\,b \\ &= 6\,(3) + 10\,(2) \\ &= 18 + 20 \\ &= 38 \text{ cm.} \end{aligned}$$

or

$$\begin{aligned} P &= 2\,(l + w) \\ &= 2\,(3\,a + 5\,b) \\ &= 2\,(9 + 10) \\ &= 2\,(19) \\ &= 38 \text{ cm.} \end{aligned}$$

(j) What is the ratio of w to l?

20. (a) Find the area of a rectangle whose dimensions are $3\,x + 5\,y$ and $2\,x + 3\,y$.

(b) Find the area in square inches if $x = 4$ in. and $y = 5$ in.

(c) A convenient form is to place the multiplication and evaluation side by side as follows:

$$
\begin{array}{rlr}
S_{\square} = l \cdot w & & \\
l = 3x + 5y & = 3(4) + 5(5) = 12 + 25 = & 37 \\
w = \underline{2x + 3y} & = 2(4) + 3(5) = 8 + 15 = & \underline{23} \\
6x^2 + 10xy & & 111 \\
\underline{+ 9xy + 15y^2} & & \underline{74} \\
\therefore S = 6x^2 + 19xy + 15y^2 & = 6(16) + 19(20) + 15(25) & \\
& = 96 + 380 + 375 & \\
& S = 851 & = 851 \\
& & \text{sq. in.}
\end{array}
$$

(d) If the two evaluations give the same number, it is proven that the multiplication is correct. Such evaluation is called *checking*, for by it any error may be checked.

(e) Correctness of the coefficients alone may be checked by letting x and y each equal 1. Thus:

$$
\begin{array}{rlr}
l = 3x + 5y & = 3 + 5 & = 8 \\
w = \underline{2x + 3y} & = 2 + 3 & = \underline{5} \\
6x^2 + 10xy & & \\
\underline{+ 9xy + 15y^2} & & \\
S = 6x^2 + 19xy + 15y^2 & = 6 + 19 + 15 & \\
& = 40 & = 40
\end{array}
$$

This method of checking does not show any errors in the letters or exponents.

21. (a) Find the areas and perimeters of rectangles having the following dimensions.

(b) Check the coefficient of Examples (1) to (8) inclusive by letting $x = y = 1$.

(c) Check completely Examples (9) to (16) inclusive by letting the first letter have a value of 2 and the second a value of 3. (Other values may be used if desired.)

	Length	Width	S	P
(1)	$a+b$	$a+b$	....	....
(2)	$2x+y$	$x+y$	....	....
(3)	$m+2n$	$m+n$	....	....
(4)	$2x+5z$	$x+3z$	....	....
(5)	$a+2x$	$a+2x$	....	....
(6)	$\frac{1}{2}a+\frac{1}{3}b$	$\frac{1}{2}a+\frac{1}{3}b$	....	....
(7)	$\frac{1}{2}x+\frac{1}{4}y$	$\frac{1}{2}x+\frac{3}{4}y$	....	....
(8)	$3c+5d$	$3c+5d$	....	....
(9)	$2a+b$	$a+2b$	....	....
(10)	$3x+2y$	$3x+2y$	....	....
(11)	$a+3b$	$a+3b$	....	....
(12)	$x+5$	$x+4$	....	....
(13)	$2x+3$	$2x+3$	....	....
(14)	$3x+5y$	$3x+7y$	....	....
(15)	$3m+2n$	$2m+n$	....	....
(16)	$2a+\frac{1}{3}b$	$a+\frac{1}{2}b$	....	....

(d) Which of these rectangles are squares?

22. (a) Examine carefully the products obtained in the previous exercise; for instance, Example 15.

$$\begin{array}{rrr}
 & \text{(1)} \quad \text{(2)} & \text{(3)} \\
3\,m & +\;2\,n & \\
2\,m & +\;\;\;n & \\
\hline
6\,m^2 & +\;4\,mn & \\
 & +\;3\,mn & +\;2\,n^2 \\
\hline
6\,m^2 & +\;7\,mn & +\;2\,n^2 \\
\text{(1)} & \text{(2)} & \text{(3)}
\end{array}$$

(b) How is the first term of the product $6\,m^2$ obtained from the first terms of the binomials?

(c) How is the third term, $2\,n^2$, obtained from the second terms of the two binomials?

(d) How is the middle term found?

Such a term is the sum of the two products obtained by multiplying crosswise.

(e) A little practice will enable one to write out the product of two binomials without showing all the work of multiplication. Thus:

(1) (3)

(1) $(3m + 2n)(2m + n) = 6m^2 + 7mn + 2n^2$

$\qquad\qquad +\qquad\qquad\qquad\qquad$ (1) $\qquad$ (2) $\qquad$ (3)

(2)

(2) $(a + 2b)(a + 3b) = a^2 + 5ab + 6b^2$ Check $a=1$

$(1 + 4)(1 + 6) = 1 + 10 + 24$ $\qquad b=2$

$(5)(7) = 35$

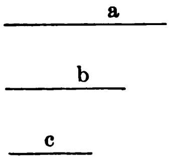
$35 = 35$

23. (a) Find the areas and perimeters of the following rectangles and check results.

(b) Which ones are squares?

	Length	Width	Area	Perimeter
(1)	$a+1$	$a+1$	....	....
(2)	$x+4$	$x+3$	....	....
(3)	$x+3$	$x+1$	....	....
(4)	$2x+3$	$2x+1$	....	....
(5)	$3n+4$	$3n+4$	....	....
(6)	$2a+3b$	$2a+b$	....	....
(7)	$3a+2b$	$a+2b$	....	....
(8)	$2s+t$	$2s+t$	....	....
(9)	$4a+3b$	$4a+5b$	....	....
(10)	$4x+b$	$4x+b$	....	....
(11)	$6x+y$	$x+6y$	....	....
(12)	$6x+5$	$x+6$	....	....
(13)	$x+9$	$x+9$	....	....
(14)	$2y+3z$	$2y+3z$	....	....
(15)	$3a+10$	$3a+10$	....	....
(16)	$p+2q$	$p+q$	....	....

24. (a) Draw three lines of different length.

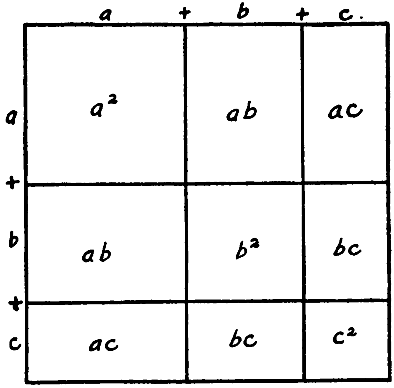


(b) Draw a line equal to the sum of the three; as, $a + b + c$.

(c) On this line construct a square.

(d) Draw lines at the points of division, as in the figure.

(e) Find the area of each part of the square.



(1) How many small parts make up the large square?

(2) How many of these parts are squares?

(3) What special position do these small squares have?

(4) How many of the parts are rectangles?

(5) How many have a and b as dimensions?

(6) How many have a and c as dimensions?

(7) How many have b and c as dimensions?

(8) What is the area of $(a + b + c)^2$?

(9) Write the area in six terms.

(10) Check with $a = 1$, $b = 2$, and $c = 3$.

25. What is the area of a square on the sum of three lines x, y, and z? Draw a figure and check.

26. (a) Three numbers a, b, and c, can be combined in three different products, ab, ac, and bc, by combining each one by every other one that follows. Besides, each can be combined with itself.

(b) What kind of figure does each of the latter combinations make? What kind does each of the former make?

(c) In how many different products can you combine four numbers, a, b, c, and d?

(d) Represent these by different lengths of lines, construct a square on the sum, and divide it into parts.

(e) How many of these parts are squares?

(f) How many are rectangles?

(g) How many different sized rectangles are there? How many of each kind?

(h) We find that

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc$ and $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2\,ab + 2\,ac + 2\,ad + 2\,bc + 2\,bd + 2\,cd$.

Thus we see that a *square constructed on the sum of any number of lines contains a square on each line plus two rectangles of each possible combination.*

27. (a) What is the area of a square constructed on the following lines?

(b) Check results.

(1) $a + 2\,b$	(6) $a + b + x + y$
(2) $a + 2\,b + c$	(7) $a + b + 2$
(3) $a + 2\,b + 3\,c$	(8) $x + y + z + 3$
(4) $2\,a + 3\,b + 4\,c$	(9) $2\,m + 3\,n + 4$
(5) $3\,x + 2\,y + z$	(10) $a + b + c + 2$

II. Finding One Dimension

1. (a) Draw a 6-inch square. What is its area?

(b) Draw a rectangle 4″ wide whose area is equal to that of the square.

(c) How do you know how long to make your rectangle?

(d) Draw another rectangle equal to the square in size with a width of 3″.

(e) If the area of a rectangle and one dimension are known, how can you find the other dimension?

(f) Write your statement as a formula:

$$l = \frac{S}{w}$$

to be read, *l equals S divided by w.*

(g) The line between the S and w is the line of the division sign (÷) in which the letters have taken the places of the two dots.

(h) 8 divided by 4 may be written in three ways:

(1) $8 \div 4$, with the division sign.

(2) $\frac{8}{4}$, as a fraction.

(3) 2, as a quotient.

(i) The division sign is not often used with letters representing numbers, but the fraction form is used instead; as $\frac{a}{b}$, which means $a \div b$.

NOTE: It is interesting to know, however, that the fraction form of showing division is about 500 years older than the division sign. The different ways of expressing division as used by various people during the last 1500 years are:

(1) From about 500 to 1200 A.D., the Hindus wrote the divisor over the dividend with no line between; as $\begin{matrix} a \\ b \end{matrix}$.

This form was found in a book written by a Hindu about 1150 and was probably used several hundred years before.

(2) About 1000, the Arabs used a straight line in one of three ways; $a - b$, a/b, or $\frac{a}{b}$.

(3) In 1631, in England, Oughtred used a dot; as, $a \cdot b$; and in 1657 a colon; as $a:b$.

(4) In 1668, in England, Pell used the division sign, as we know it and as it is used today in English speaking countries.

(j) Any fraction is an indicated division; as, $\frac{8}{4}$, $\frac{2}{3}$, $\frac{a}{b}$, $\frac{a+b}{b}$, $\frac{x^2}{x}$, $\frac{xy}{x}$.

(k) Sometimes the division may be performed and one number be obtained as a quotient; as,

(1) $\frac{8}{4} = 2$ or $\frac{8}{4} = \frac{\cancel{2} \cdot \cancel{2} \cdot 2}{\cancel{2} \cdot \cancel{2}} = 2$

(2) $\frac{x^2}{x} = x$ or $\frac{x^2}{x} = \frac{\cancel{x} \cdot x}{\cancel{x}} = x$

(3) $\frac{xy}{x} = y$ or $\frac{xy}{x} = \frac{\cancel{x} \cdot y}{\cancel{x}} = y$

(l) With other numbers the division cannot be performed and we use the expression in the form of a fraction; as, $\frac{2}{3}$, $\frac{15}{16}$, $\frac{a}{b}$, $\frac{a+b}{b}$, $\frac{x}{y}$, $\frac{x}{x+y}$.

2. (a) Write the formula for the width of a rectangle.

(b) Translate the formula into English.

3. (a) In each of the following rectangles, the area and one dimension are given.

(b) Find the other dimension.

	$S\square$	l	w
(1)	144 sq. in.	.....	9 in.
(2)	289 sq. ft.	17 ft.	.....
(3)	$30\frac{1}{4}$ sq. yd.	$5\frac{1}{2}$ yd.	.....
(4)	288 sq. ft.	.....	12 ft.
(5)	$15\frac{1}{2}$ sq. in.	.....	$3\frac{3}{4}$ in.
(6)	$12\frac{1}{2}$ sq. ft.	.....	$1\frac{1}{4}$ ft.
(7)	289 sq. rd.	.....	$12\frac{3}{4}$ rd.
(8)	256 sq. in.	32 in.	.....
(9)	4.41 sq. in.	6.3 in.	.....
(10)	72.8 sq. ft.	.....	3.8 ft.

4. If a rectangle contains $12\,x^2$ sq. ft. and is $4\,x$ feet long, how wide is it?

(a) *Solution.*

$$\begin{aligned} w &= \frac{S}{l} \\ &= \frac{12\,x^2}{4\,x} \\ &= \frac{\cancel{2}\cdot\cancel{2}\cdot 3\cdot\cancel{x}\cdot x}{\cancel{2}\cdot\cancel{2}\cdot\cancel{x}} \\ &= 3\,x \end{aligned}$$

$\therefore \square$ is $3\,x$ ft. wide.

(b) *Explanation:*

(1) The fraction $\dfrac{12\,x^2}{4\,x}$ may be reduced to lowest terms by dividing both the numerator and denominator by all of the common factors, just as $\frac{18}{24}$ is reduced to $\frac{3}{4}$.

$$\frac{18}{24} = \frac{\cancel{2}\cdot 3\cdot\cancel{3}}{\cancel{2}\cdot 2\cdot 2\cdot\cancel{3}} = \frac{3}{4}$$

(2) Or, the indicated division may be solved as a short division problem as, $6\overline{)18}$ with quotient 3.

Then $\dfrac{12\,x^2}{4\,x}$ is $4\,x\overline{)12\,x^2}$ with quotient $3\,x$.

(3) To find the quotient $3\,x$, the coefficient 12 is divided by the 4 as in arithmetic.

(4) But dividing the x^2 by the x is quite different. We found that

$$3 \cdot 3 = 3^2 \quad \therefore \quad \frac{3^2}{3} = 3$$

$$\text{and } x \cdot x = x^2 \quad \therefore \quad \frac{x^2}{x} = x$$

(5) We say that x^2 is the second power of x and x^3 is the third power of x, and write the little 2 and 3 as exponents to show how many factors have been used.

But we do not use the exponent 1, as x^1 to show that x is the first power, although it is always understood. In finding the product of $x \times x$ or $x^1 \times x^1$ we *add* the two 1's to get the exponent of x^2.

Likewise $x \cdot x \cdot x = x^1 \cdot x^1 \cdot x^1 = x^{1+1+1} = x^3$

and $x^2 \cdot x = x^2 \cdot x^1 = x^{2+1} = x^3$.

(6) Since we *add* the exponents of *like* letter factors to find the product, we must do the *opposite* to find the quotient; that is, we must *subtract* the exponent of the divisor from the exponent of the dividend to find that of the quotient.

Therefore, $x^2 \div x = [x^2 \div x^1 = x^{2-1} = x^1] = x$

and $x^3 \div x = [x^3 \div x^1 = x^{3-1}] = x^2$

and $x^3 \div x^2 = [x^{3-2} = x^1] = x$.

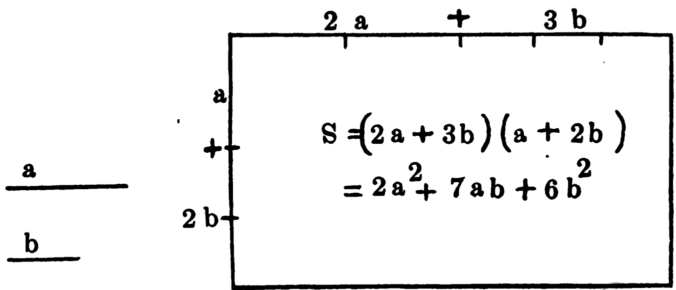
NOTE: The parts in brackets are given here only for the sake of explanation.

5. (a) Find the missing measurement of the following rectangles from the two that are given.

(b) Use the formulas.

	Area	Length	Width
(1)	$24x^2$	.....	$4x$
(2)	$24x^2$	$8x$	.....
(3)	$108a^2$	$12a$	.....
(4)	$117y^2$	.....	$3y$
(5)	.....	$12b$	$5\frac{1}{2}b$
(6)	$78t$	.....	$6t$
(7)	$12xy$	.....	$2y$
(8)	$25ab$	$5a$	.....
(9)	$69\cdot mn$	.....	$3m$
(10)	$72ax$	$36a$	.....
(11)	$72ax$	.....	$2a$
(12)	$72ax$	.....	$8x$
(13)	$72ax$	.....	$6a$
(14)	.....	$18a$	$4x$
(15)	.....	$a+1$	a
(16)	$a(a+b)$	.....	a
(17)	.....	$x+y$	3
(18)	$6(a+x)$	$a+x$	.....
(19)	.....	$3(m+n)$	$4m$
(20)	$21a(a+c)$	$3(a+c)$	....

6. Draw a rectangle whose length is $2a + 3b$ and whose width is $a + 2b$. Find its area.



Given a ▭, with $l = 2a + 3b$
and $w = a + 2b$

Discussion.

(a) We may write the area in two ways:

(1) $S = (2a + 3b)(a + 2b)$, expressed in factors.

(2) $S = 2a^2 + 7ab + 6b^2$, expressed as a product.

(b) If the area is given in factors and one dimension is known, then the other dimension is easily found.

$$l = \frac{S}{w} = \frac{(2a + 3b)(a + 2b)}{a + 2b}$$
$$= 2a + 3b$$

(c) Remember that in this area *each* factor has two parts or terms and that the factor as a whole must be used and not the parts taken separately.

(d) If the area is expressed as a product, then the second dimension must be found by long division.

(e) Short division is used if the divisor is small enough, whether it contains numbers or letters; as,

$$3\,\overline{)18}\quad 6 \qquad\qquad 3a\,\overline{)18a^2}\quad 6a \qquad\qquad 3a\,\overline{)18a^2 + 6a}\quad 6a + 2$$

(f) Long division is used if the divisor contains more than one term.

(g) Compare with long division in arithmetic.

$$\begin{array}{r|l} 31\,\overline{)961}\ \ 31 & \\ \underline{93} & \\ 31 & \\ \underline{31} & \end{array} \quad \text{or} \quad \begin{array}{r} 30 + 1 = 31 \\ 30 + 1\,\overline{)900 + 60 + 1} \\ \underline{900 + 30} \\ 30 + 1 \\ \underline{30 + 1} \end{array}$$

The second illustration is just like the first except that the numbers are expressed with units, tens,

and hundreds separated. The second form is always used in long division of numbers expressed with letters.

If t represents tens, then t^2 will represent hundreds for

$100 = 10^2 = t^2$; and $31 = 30 + 1 = 3t + 1$ and

$961 = 900 + 60 + 1 = 9t^2 + 6t + 1$

$$\begin{array}{r|l}
 & \underline{3t+1} = 30 + 1 = 31 \\
\underline{3t+1} & 9t^2 + 6t + 1 \\
 & \underline{9t^2 + 3t} \\
 & \qquad 3t + 1 \\
 & \qquad \underline{3t + 1}
\end{array}$$

Solution.

Find the length of the rectangle by long division.

Given $S = 2a^2 + 7ab + 6b^2$

and $w = a + 2b$

$$l = \frac{S}{w} = \frac{2a^2 + 7ab + 6b^2}{a + 2b}$$

$$\begin{array}{r|l}
 & \underline{\qquad 2a + 3b} \\
\underline{a + 2b} & 2a^2 + 7ab + 6b^2 \\
 & \underline{2a^2 + 4ab} \\
 & \qquad 3ab + 6b^2 \\
 & \qquad \underline{3ab + 6b^2}
\end{array}$$

$\therefore\ l = 2a + 3b$

Explanation.

(a) Since a is the first term of the divisor, the term of the dividend having the highest power of a must come first; the next lower one, second; and the term without a must come last.

(b) By dividing a into $2a^2$, we get $2a$, which is the first term of the quotient.

(c) By multiplying both terms of the divisor by $2a$, we get $2a^2 + 4ab$, which is to be subtracted from the dividend.

(d) The second term of the quotient $3b$ is found by dividing a into $3ab$.

(e) Multiply $a + 2b$ by $3b$ and subtract the product from $3ab + 6b^2$. There is no remainder.

Problems:

(a) Given $S = 2a^2 + 7ab + 6b^2$
$l = 2a + 3b$

To find w.

$$w = \frac{S}{l} = \frac{2a^2 + 7ab + 6b^2}{2a + 3b}$$

$$\begin{array}{r|l} & a + 2b \\ \hline 2a + 3b & 2a^2 + 7ab + 6b^2 \\ & \underline{2a^2 + 3ab} \\ & \qquad 4ab + 6b^2 \\ & \qquad \underline{4ab + 6b^2} \end{array} \qquad \therefore w = a + 2b$$

(b) Given $S = 6x^2 + 19xy + 15y^2$
$w = 2x + 3y$

To find l.

$$l = \frac{S}{w} = \frac{6x^2 + 19xy + 15y^2}{2x + 3y}$$

$$\begin{array}{r|l} & 3x + 5y \\ \hline 2x + 3y & 6x^2 + 19xy + 15y^2 \\ & \underline{6x^2 + 9xy} \\ & \qquad 10xy + 15y^2 \\ & \qquad \underline{10xy + 15y^2} \end{array} \qquad \therefore l = 3x + 5y$$

(c) *Proof:* (1) Proof by multiplication.

$$S = l \cdot w$$
$$6x^2 + 19xy + 15y^2 = (3x + 5y)(2x + 3y)$$
$$= 6x^2 + 19xy + 15y^2$$

Product by inspection.

$$\begin{array}{l} 3x \;\; + 5y \\ \underline{2x \;\; + 3y} \\ 6x^2 + 10xy \\ \underline{\quad\;\; + \;\; 9xy + 15y^2} \\ 6x^2 + 19xy + 15y^2 \end{array}$$

Product by long multiplication.

(2) Proof by checking.

Let $a = 2$ and $b = 3$.

$$\frac{6x^2 + 19xy + 15y^2}{2x + 3y} = 3x + 5y$$

$$\frac{6(2)^2 + 19(2)(3) + 15(3)^2}{2(2) + 3(3)} = 3(2) + 5(3)$$

$$\frac{24 + 114 + 135}{4 + 9} = 6 + 15$$

$$\frac{273}{13} = 21$$

$$21 = 21$$

The coefficients alone may be checked by substituting $x = y = 1$.

$$\frac{6x^2 + 19xy + 15y^2}{2x + 3y} = 3x + 5y$$

$$\frac{6 + 19 + 15}{2 + 3} = 3 + 5$$

$$\frac{40}{5} = 8$$

$$8 = 8$$

(3) Proof by division. Use $w = \frac{S}{l}$.

$$\begin{array}{r|l} & \quad\quad\quad\;\; 2x + 3y \\ \hline 3x + 5y & 6x^2 + 19xy + 15y^2 \\ & \underline{6x^2 + 10xy} \\ & \quad\quad\;\; 9xy + 15y^2 \\ & \quad\quad\;\; \underline{9xy + 15y^2} \end{array}$$

7. (a) Find the missing dimension of each of the following rectangles. (b) Prove your results correct.

No.	Area	Length	Width
(1)	$2x^2 + 22x + 48$	$2x + 6$	....
(2)	$4m^2 + 21mn + 27n^2$	....	$m + 3n$
(3)	....	$5a + 16b$	$a + 4b$
(4)	....	$3x + 4y$	$x + 2y$
(5)	$20a^2 + 58ab + 42b^2$	$5a + 7b$	....
(6)	$3c^2 + 14cd + 8d^2$	....	$c + 4d$
(7)	$2x^2 + 7xy + 5y^2$	....	$x + y$
(8)	$6n^2 + 23nk + 15k^2$	$6n + 5k$	....
(9)	....	$8x + 3y$	$4x + 2y$
(10)	....	$9a + 6b$	$3a + 2b$
(11)	$x^2 + 14x + 48$	$x + 8$	....
(12)	$3t^2 + 11tu + 10u^2$	....	$t + 2u$
(13)	....	$4r + 3s$	$3r + 4s$
(14)	$2x^2 + 19x + 42$	$2x + 7$	....
(15)	$14a^2 + 27ab + 9b^2$	....	$2a + 3b$
(16)	....	$8a + 3x$	$4a + 7x$
(17)	$27x^2 + 15xy + 2y^2$	$9x + 2y$	....
(18)	$15a^2 + 37ab + 20b^2$	....	$3a + 5b$
(19)	....	$5e + 4f$	$3e + 6f$
(20)	$32m^2 + 16mn + 22n^2$	$8m + 2n$	....
(21)	$9a^2 + 24ab + 16b^2$	$3a + 4b$	....
(22)	$15x^2 + 26xy + 8y^2$	....	$3x + 4y$
(23)	....	$\frac{1}{2}a + \frac{1}{3}b$	$\frac{1}{2}a + \frac{1}{3}b$
(24)	$a^2 + 10a + 21$	$a + 7$	....
(25)	$\frac{1}{9}x^2 + \frac{8}{3}x + 16$	$\frac{1}{3}x + 4$	....
(26)	$81a^2 + 9a + \frac{1}{4}$	....	$9a + \frac{1}{2}$
(27)	....	$3a + 5b + 6c$	$a + 2b + c$
(28)	$36a^2 + 60ax + 25x^2$	$6a + 5x$	....
(29)	$14r^2 + 81rt + 90t^2$	$7r + 30t$	....
(30)	$.01a^2 + .04ab + .04b^2$	....	$.1a + .2b$
(31)	....	$.2x + .3y$	$.2x + .3y$
(32)	$8x^2 + 76xk + 140k^2$	....	$x + 7k$
(33)	....	$13W + 3F$	$2W + 7F$
(34)	$x^2 + 24x + 128$	....	$x + 8$
(35)	....	$14c + 5m$	$2c + 3m$
(36)	$25x^2 + 56xy + 21y^2$	$5x + 7y$	....

(c) Which of the above rectangles are squares?

III. Problems of Finding Dimensions from Areas

1. If you know only the area of a rectangle, can you find what the dimensions are?

2. (a) If a rectangular flower bed contains 36 sq. ft., you cannot tell whether it is

(1) $1' \times 36'$
(2) $2' \times 18'$
(3) $3' \times 12'$
(4) $4' \times 9'$
(5) $6' \times 6'$
(6) or other fractional dimensions, as $4\frac{1}{2}' \times 8'$.

(b) But if the area is expressed with letters, you can find the two dimensions by finding its two factors. Usually only one pair of factors can be found.

3. (a) If $a^2 + 3\,ab$ is the area of a rectangle, we can easily see that a must be one of the dimensions, for a is found in each term.

(b) By dividing the area, $a^2 + 3\,ab$, by one dimension, a, we find the other to be $a + 3\,b$.

$$l = \frac{S}{w} = \frac{a^2 + 3\,ab}{a} = a + 3\,b$$

or

$$a\overline{\big)\,a^2 + 3\,ab}\quad\text{quotient } a + 3\,b$$

(c) Instead of writing the problem in the form of a fraction or division, it is customary to write it in the form of two factors, or like the formula.

$$S = w \cdot l$$
$$a^2 + 3\,ab = a\,(a + 3\,b)$$

4. (a) If the area is $2\,a^2 + 6\,ab$, we see that each term contains 2 as well as a, so we take out $2\,a$.

$$2\,a^2 + 6\,ab = 2\,a\,(a + 3\,b)$$

(b) What is the length of such a rectangle?
What is the width?

(c) Find its length, width, and area

when $a = 2$ ft.
and $b = 3$ ft.

(d) Find these values when a is 10″ and b is 2″.

(e) What is the ratio of the width to the length in each rectangle?

5. (a) Find the dimensions of the following rectangles from their areas.

(b) Check results.

No.	Area	No.	Area
(1)	$xy =$	(19)	$ax + bx + cx =$
(2)	$2\,x^2 + 3\,xy =$	(20)	$4\,by + 6\,cy + 10\,dy =$
(3)	$2\,x^2 + 4\,xy =$	(21)	$3\,ab + 6\,ax + 6\,ay =$
(4)	$5\,x^2 + 10\,x =$	(22)	$\frac{1}{2}\,ax + \frac{1}{4}\,bx + \frac{1}{6}\,cx =$
(5)	$6\,a^2 + 9 =$	(23)	$.2\,x^2 + .4\,ax =$
(6)	$2\,m^2 + 6\,mn =$	(24)	$\frac{1}{5}\,b^2 + \frac{1}{15}\,b =$
(7)	$10\,c^2 + 15\,cd =$	(25)	$(a+3)^2 + 4\,(a+3) =$
(8)	$21\,m^2 + 14\,mn =$	(26)	$(m+n)^2 + 5\,(m+n) =$
(9)	$3\,x^2 + xz =$	(27)	$a\,(m+n) + b\,(m+n) =$
(10)	$.5\,x^2 + .25\,xy =$	(28)	$\frac{2}{3}\,a^2 + \frac{8}{9}\,ab =$
(11)	$\frac{1}{2}\,a^2 + \frac{1}{4}\,a =$	(29)	$\frac{1}{3}ab + \frac{1}{6}\,b^2 + \frac{1}{9}\,bc =$
(12)	$\frac{1}{2}\,x^2 + \frac{1}{4}\,xy =$	(30)	$(c+2)\,(c+2) + (a+3)\,(c+2) =$
(13)	$ab + ac + ad =$	(31)	$(m+n)\,(m+n) + 3\,(m+n) =$
(14)	$xy + y^2 + yz =$	(32)	$\frac{1}{2}\,hB + \frac{1}{2}\,hb =$
(15)	$\frac{1}{3}\,b^2 + \frac{1}{9}\,b =$	(33)	$(a+5)\,(a+1) + (a+5)\,(a+3) =$
(16)	$3\,ax + 6\,x^2 + 9\,bx =$	(34)	$x^2yz + xy^2z + xyz^2 =$
(17)	$3\,(a+b)^2 + 6\,(a+b) =$	(35)	$2\,a^2bc + 4\,ab^2c + 6\,abc^2 =$
(18)	$(x+y)^2 + 2\,(x+y) =$	(36)	$.5\,a^2b^2 + a^2c^2 + 1.5\,a^2d^2 =$

6. (a) We have learned how to find the dimensions of a rectangle from its area when each of its terms has a common factor.

(b) But there is no such common factor in

$$6\,x^2 + 19\,xy + 15\,y^2$$

although that is the area of a rectangle.

(c) On page 115, we learned how to write the product of two binomials. Reread that page.

(d) To find the dimensions from the area we must reverse that process.

$$6\,x^2 + 19\,xy + 15\,y^2 = (\quad ? \quad)(\quad ? \quad)$$

$$\begin{array}{r} 6\,x + 5\,y \\ x + 3\,y \\ \hline (6\,x^2) + 5\,xy \qquad\qquad \\ + 18\,xy + (15\,y^2) \\ \hline (6\,x^2) + 23\,xy + (15\,y^2) \end{array}$$

$$\begin{array}{r} 3\,x + 5\,y \\ 2\,x + 3\,y \\ \hline 10\,xy \\ 9\,xy \\ \hline 19\,xy \end{array}$$

(1) The $6\,x^2$ may come from $6\,x \times x$ or $3\,x \times 2\,x$ and $15\,y^2$ may come from $5\,y \times 3\,y$ or $15\,y \times y$.

(2) Try these in all possible combinations, until the set is found that gives the right middle term of $19\,xy$ when multiplied crosswise and added.

(3) The first set tried gave $23\,xy$ as the middle term and is therefore incorrect.

(4) The second gave $19\,xy$; therefore $3\,x + 5\,y$ and $2\,x + 3\,y$ are the length and width of the rectangle, respectively.

$$6\,x^2 + 19\,xy + 15\,y^2 = (3\,x + 5\,y)(2\,x + 3\,y)$$
$$S = l \times w$$

(5) Several other combinations may be tried before the correct one is found.

(6) Try these other combinations to see if any gives the correct cross product.

$$\begin{array}{ccc} \text{A} & \text{B} & \text{C} \\ 6\,x + y & 2\,x + 15\,y & 3\,x + 3\,y \\ \underline{x + 15\,y} & \underline{3\,x + y} & \underline{2\,x + 5\,y} \end{array}$$

$$\begin{array}{ccc} \text{D} & \text{E} & \text{F} \\ 6\,x + 3\,y & 6\,x + 15\,y & 3\,x + 15\,y \\ \underline{x + 5\,y} & \underline{x + y} & \underline{2\,x + y} \end{array}$$

(7) If one had to try so many different combinations before finding the right set, it would be a tedious process, but just a glance at any of these last six shows that they are all impossible.

In set A, $6\,x \times 15\,y = 90\,xy$, which is entirely too large. Likewise, in set B, $3\,x \times 15\,y = 45\,xy$.

In the last four, there is still another way to know that these cross products need not be tried. In each group one of the dimensions can be separated again into two factors by dividing by 3; as, $3\,x + 3\,y = 3\,(x + y)$ and $6\,x + 3\,y = 3\,(2\,x + y)$.

If 3 can be taken out of one of the two factors of the trinomial, it can be taken out of the trinomial itself and should be done first. Thus:

$$\begin{aligned} 6\,x^2 + 21\,xy + 15\,y^2 &= 3\,(2\,x^2 + 7\,xy + 5\,y^2) \\ &= 3\,(2\,x + 5\,y)(x + y) \end{aligned}$$

Since it is impossible to take out 3 or any other common factor from the given area of $6\,x^2 + 19\,xy + 15\,y^2$, it will be impossible to take such a factor out of either dimension. Therefore none of the last four need

be tried. With a little practice one can easily find the two factors by inspection, that is, by looking at the problem.

7. (a) Find the dimensions of the following rectangles from the given areas. Check each.

Given areas are:

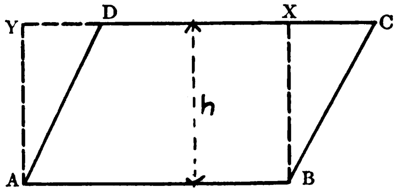
(1) $a^2 + 8a + 15$
(2) $m^2 + 12mn + 36n^2$
(3) $x^2 + 10x + 24$
(4) $3a^2 + 8a + 4$
(5) $4b^2 + 8b + 3$
(6) $8y^2 + 22y + 12$
(7) $d^2 + 2d + 1$
(8) $c^2 + 2cd + d^2$
(9) $x^2 + 11ax + 30a^2$
(10) $49x^2 + 14bx + b^2$
(11) $6x^2 + 17x + 12$
(12) $c^2 + 16c + 63$
(13) $49a^2 + 7ab + 25b^2$
(14) $4m^2 + 16m + 7$
(15) $4a^2 + 8a + 3$
(16) $9x^2 + 9xy + 2y^2$
(17) $4a^2 + 4a + 1$
(18) $4x^2 + 4xy + y^2$
(19) $2a^2 + 3ay + y^2$
(20) $2x^2 + 5xy + 2y^2$
(21) $m^2 + 4mn + 4n^2$
(22) $2a^2 + 5a + 3$
(23) $3b^2 + 13bx + 12x^2$
(24) $4c^2 + 16c + 15$
(25) $4a^2 + 23a + 15$
(26) $y^2 + 10y + 25$
(27) $m^2 + 16m + 48$
(28) $d^2 + 26d + 169$
(29) $5^2 + 2(5) + 1$
(30) $20^2 + 6(20) + 3$
(31) $10^2 + 15(10) + 56$
(32) $20^2 + 10(20) + 21$

(b) Which of these rectangles are squares?

CHAPTER NINE

REVIEW PROBLEMS

A. PROBLEMS ON PARALLELOGRAMS



1. (a) Measure the ∠s of the given ▱, the sides and height or altitude.

(b) Compare $\angle A$ with $\angle C$; $\angle B$ with $\angle D$.

(c) What is the sum of $\angle A + \angle B$?
What kind of ∠s are they?

(d) How many pairs of supplementary ∠s can you find?

(e) What is the sum of $\angle A + \angle B + \angle C + \angle D$?
How does this sum compare with the sum of the ∠s of a rectangle?
Compare it with the sum of the ∠s of a △.

(f) Draw several different sized ▱s and find the sum of their angles.

(g) Draw several irregular quadrilaterals and find the sum of their angles.

(h) To find all the ∠s of a ▱, how many do you have to measure?

(i) What is the ratio of AD to AB?

(j) What is the ratio of the height to the base?

(k) What is the area and perimeter of $ABCD$?

(1) Draw the ⊥s AY and BX. What kind of quadrilateral is $ABXY$? What is its area? Its perimeter?

2. Draw a parallelogram with two adjacent sides $2\frac{3}{4}''$ and $1\frac{1}{2}''$ respectively. The ∠ between these sides is 55°.

(a) What is ratio of the two sides?

(b) Draw the height and measure it to the nearest tenth of an inch.

(c) Find the area.

3. (a) Two city streets cross each other at an angle of 110°. The block made by them and two other parallel streets measures 240 feet on one street and 320 feet on the other.

(b) Draw a diagram of the block on a scale of 100′ to 1″.

4. (a) Two sides of a parallelogram are x and y. Their ratio is $\frac{3}{4}$. The included ∠ is 45°.

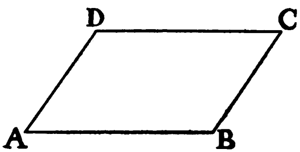
(b) Draw the ▱. Find the perimeter in terms of the line y.

(c) Draw the height. How does it compare with the base y?

(d) If the ratio of the height to the base is .53, what is the area?

(e) Find the perimeter and area if y = 12 cm.

5. (a) The parallelogram is drawn to a scale of $\frac{1}{8}''$ to a foot.



(b) Find its dimensions.

(c) Measure its angles.

(d) Find the ratio of its sides. Of ∠ A to ∠ B.

(e) Draw its altitude and measure to the nearest tenth of an inch.

(f) Compute its perimeter and its area.

6. (a) A cement pavement $3\frac{1}{2}$ ft. wide surrounds a rectangular city block $200' \times 300'$, measured inside the pavement.
   (b) How many square yards are in the pavement?
7. (a) The city block is in the shape of a parallelogram 300 and 400 feet on the sides, which make an angle of 55°.
   (b) Draw to a scale and find its area.
   (c) Find the area of a 4 ft. pavement surrounding this block.

B. PROBLEMS ON TRIANGLES

1. Draw triangles from the following data.
   Measure the sides and angles not given.

(a) $AB = 2\frac{1}{2}''$	$AC = 1\frac{3}{8}''$	$\angle A = 55°$
(b) $AB = 3\frac{3}{4}''$	$\angle A = 35°$	$\angle B = 65°$
(c) $AB = 1\frac{1}{2}''$	$AC = 2\frac{1}{4}''$	$BC = 2\frac{3}{4}''$

2. (a) Draw a rt. $\triangle$ whose perpendicular sides are $1\frac{1}{4}$ and 3 inches respectively.
   (b) Compute the length of the hypotenuse.
   (c) Measure it to verify your computation.
   (d) How many degrees are there in each of the acute $\angle$s?
3. (a) Draw a rt. $\triangle$ whose base is 12 centimeters and hypotenuse is 15 centimeters.
   (b) Measure the acute angles and the third side.
   (c) What is the Pythagorean Theorem?
   (d) This formula may be used to find either leg of a right triangle as well as the hypotenuse.

We know that $3 + 4 = 7$

$$\therefore 3 = 7 - 4$$

$$\text{and } 4 = 7 - 3$$

If a number is subtracted from both sides of an equation the remainders are still equal. In the

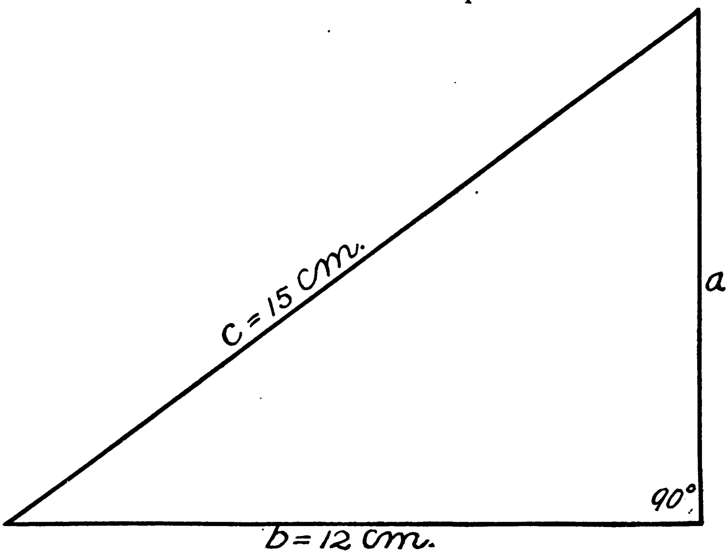
same way we may apply this axiom to the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$
$$a^2 = c^2 - b^2$$
and $b^2 = c^2 - a^2$

Equals subtracted from equals give equals.

$$a = \sqrt{c^2 - b^2}$$
and $b = \sqrt{c^2 - a^2}$

The square roots of the two sides of an equation are equal.



Solution. Given $c = 15$ cm.
$b = 12$ cm.

To find a.

$$\begin{aligned} a^2 &= c^2 - b^2 \\ &= (15)^2 - (12)^2 \\ &= 225 - 144 \\ &= 81 \\ \therefore a &= \sqrt{81} \\ &= 9 \text{ cm.} \end{aligned}$$

4. (a) Draw a rt. △ whose base is 12 inches and hypotenuse is 13 inches.
(b) Compute the length of the third side.
(c) Measure it to verify your computation.

5. A telegraph pole 50 feet high is to be steadied by a wire fastened to the pole 30 feet above the ground and to a stake in the ground 40 feet from the base of the pole. How long must the wire be if 1 foot is allowed for fastening it to the stake and 3 feet for fastening it to the pole?

6. (a) Draw a rt. △ whose base is 3″ and hypotenuse is $3\frac{3}{4}$″.
(b) Measure the acute ∠s.
(c) Compute the third side.

Solution.

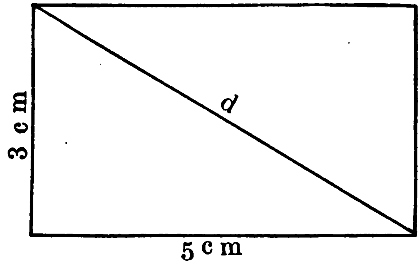
Given $c = 3\frac{3}{4}''$

$b = 3''$

To find a.

$$\begin{aligned} a^2 &= c^2 - b^2 \\ &= \left(\tfrac{15}{4}\right)^2 - 3^2 \\ &= \tfrac{225}{16} - 9 \\ &= \tfrac{225}{16} - \tfrac{144}{16} \\ &= \tfrac{81}{16} \\ \therefore a &= \sqrt{\tfrac{81}{16}} \qquad \text{Why?} \\ &= \tfrac{9}{4} \\ &= 2\tfrac{1}{4}'' \end{aligned}$$

7. (a) Draw the diagonal in a rectangle 5 centimeters long and 3 centimeters wide.



(b) Measure the diagonal to the nearest millimeter.

(c) Compute the length of the diagonal by the Pythagorean Theorem and compare results.

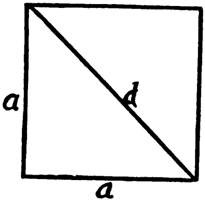
8. (a) Draw a square with a diagonal.

(b) Then $d^2 = a^2 + a^2$ Why?

$d^2 = 2\,a^2$

$\sqrt{d^2} = \sqrt{2\,a^2}$ Why?

$d = a\sqrt{2}$



Since d represents the diagonal and a represents the side of the square, we find that the diagonal of a square is equal to the side multiplied by $\sqrt{2}$ or by 1.414 +. In other words, the diagonal is about $1\frac{2}{5}$ times the side of a square.

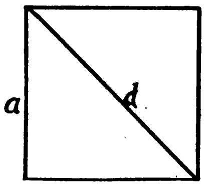
9. If the side of a square is 12 inches, find the diagonal.

Solution No. 1.

$$\begin{aligned} d^2 &= a^2 + a^2 && \text{Pythag. Th.} \\ &= 12^2 + 12^2 \\ &= 144 + 144 \\ &= 288 \\ \therefore d &= \sqrt{288} \\ &= 16.97 \end{aligned}$$

Sq. Root Process

	$\overline{2}'\ \overline{88}.\overline{00}'\ \overline{00}$ | 16.97
	1
26	188
	156
329	3200
	2961
3387	23900
	23709



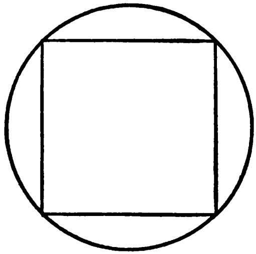
a = 12″

Solution No. 2.

$$\begin{aligned} d_{\square} &= a\sqrt{2} && \text{Formula from Ex. 8} \\ &= 12 \times 1.4142 \\ &= 16.9704 \end{aligned}$$

10. A baseball diamond is a square 90 ft. on a side. What is the distance from first to third base?

11. (a) In machine shops, "stock" comes in rods of different sizes with circular ends. All square rods must be cut from such round stock.



(b) What must be the diameter of the round stock used to cut a square rod $1\frac{1}{2}''$ on a side? A $2''$ rod? A $2\frac{1}{4}''$ rod?

12. A window is 36 feet above ground. How far out from the foot of the wall must a 45 ft. ladder be placed to just reach the window?

13. The side of one square is 32 in. and that of another is 17 in. What is the side of a square equal to the sum of these squares?

14. If the sum of two squares is 26 square inches and one of the squares is 14 square inches, what is the side of the other square?

15. A shelf 1 foot wide is $5\frac{1}{2}$ ft. from the floor. The foot of a ladder is placed $5\frac{1}{2}$ ft. from the wall. How long must the ladder be to reach the shelf?

16. A ladder 42 ft. long can be so placed that it will reach a window 31 ft. above the ground on one side of the street, and by tipping it back without moving its foot, it will reach a window 19 ft. above the ground on the other side. Find the width of the street.

17. In a right triangle, $a = 13.6''$, $b = 16.9''$. Find c.

18. An equilateral triangle is 20 inches on a side. Find its altitude.

19. The base of an isosceles triangle is 136 ft. Its altitude is 60 ft. How long is its side? What is its perimeter?

20. (a) Draw an equilateral triangle ABC and bisect $\angle C$ by the line CD.

(b) How does CD cut AB?

(c) What kind of an angle is $\angle ADC$?

(d) Cut along the line CD and describe the two parts.

(e) Measure the $\measuredangle$ of $\triangle ADC$.

(f) How does the shortest side compare with the hypotenuse?

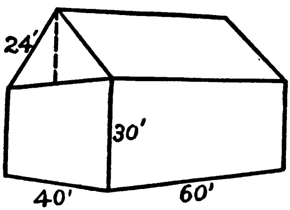
(g) Does this relation hold true between the shortest side and the hypotenuse of every 30°–60° right $\triangle$?

21. How do the two perpendicular sides of a 45°–45° right triangle compare?

22. If the vertex angle of an isosceles triangle is 80°, what is the size of each angle at the base?

23. Construct an isosceles triangle with a base $2\frac{1}{2}''$, whose vertex angle is 70°.

24. (a) A barn is 60 ft. long, 40 ft. wide, and 30 ft. high. The sloping edge of the roof is 24 ft.



(b) Find the area of each gable end.

(c) Find the total lateral area.

(d) Find the area of the roof.

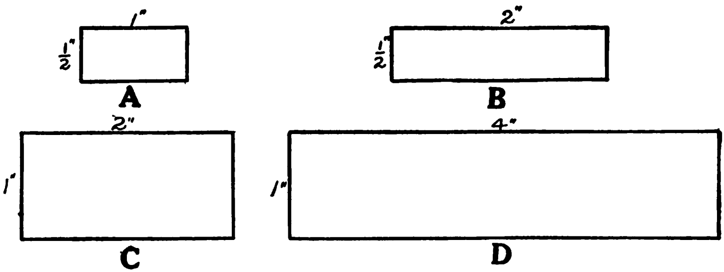
CHAPTER TEN

SIMILAR FIGURES

A. SIMILAR RECTANGLES

1. Draw four rectangles as follows:

A. $\frac{1}{2}'' \times 1''$
B. $\frac{1}{2}'' \times 2''$
C. $1'' \times 2''$
D. $1'' \times 4''$



2. What is the ratio of the height to the base in each of the four $\boxed{s}$?

3. Which ones have the same ratio?

4. Which ones may be considered small maps of another?

5. Draw diagonals in Figs. *A* and *C*.

Measure them and find their ratio.

6. Cut out Fig. *A* and place on Fig. *C* so that the centers are together and the diagonals take the same direction. What position do the sides of Fig. *A* take compared with those of Fig. *C*?

7. Draw a fifth rectangle *E*, $1\frac{1}{2}''$ by $3''$. Draw the diagonals and place Figs. *A*, *C*, and *E* together.

8. Try to place Fig. *A* with Fig. *B* or Fig. *D*.

9. Which of these are alike in shape?

Figures that are alike in shape are called *similar figures*.

B. SIMILAR PARALLELOGRAMS

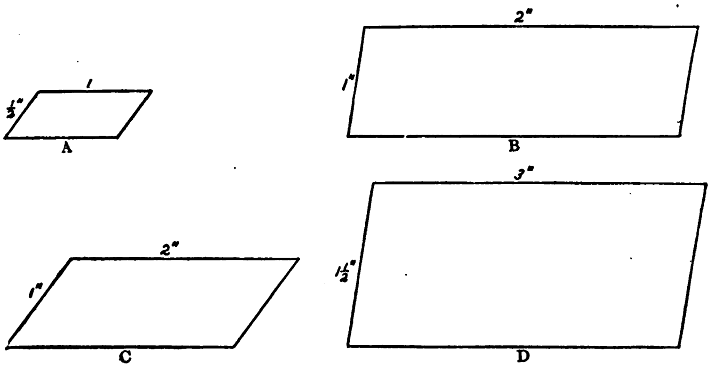
1. (a) Draw four parallelograms:

A. Sides $\frac{1}{2}''$ and $1''$ and included $\angle$ 50°

B. Sides $1''$ and $2''$ and included $\angle$ 80°

C. Sides $1''$ and $2''$ and included $\angle$ 50°

D. Sides $1\frac{1}{2}''$ and $3''$ and included $\angle$ 80°



(b) What is the ratio of the side to the base in each ▱?

(c) Is Fig. *B* equal to Fig. *C*? Why or why not?

(d) Draw the diagonals, cut out Figs. *A* and *B* and try to fit them on each other and the others.

(e) Which ones are similar?

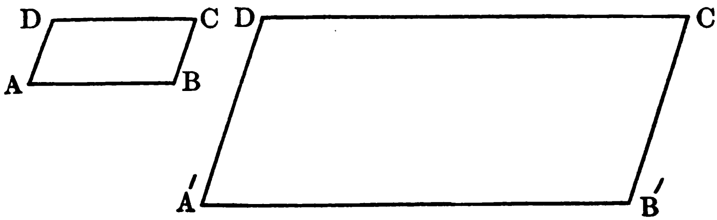
(f) Compare the ∠s in the pairs of similar figures.

(g) In similar figures,

(1) What must be true about their respective ∠s, taken in pairs?

(2) What must be true about the ratio of their respective sides?

2. (a) Draw a parallelogram similar to $ABCD$ whose corresponding sides are 3 times as long.



(b) We sometimes letter two similar figures alike except that we put a little accent mark (′) beside the letters of the second figure. A' is read *A prime.*

The ▱ $A'B'C'D'$ is read *the parallelogram A prime, B prime, C prime, D prime.*

(c) $\angle A'$ is made equal to $\angle A$.
$\angle B'$ is made equal to $\angle B$.

(d) We call $\angle A$ and $\angle A'$ *corresponding angles* in similar figures, or *homologous angles.*

(e) What side corresponds to *AB?* to *BC?* to *CD?*

(f) What $\angle$ corresponds to $\angle$ *D?*

(g) What is the ratio of AB to *A′B′?* of BC to *B′C′?*

(h) In similar figures what must be true of all corresponding or homologous angles?

(i) The word similar is used so much that it is convenient to have a symbol for it.

The double curve ($\sim$) means *is* (or *are*) *similar to.*

▱ $ABCD \sim$ ▱ $A'B'C'D'$ means *the parallelo-*

gram ABCD is similar to the parallelogram A prime, B prime, C prime, D prime.

C. SOME PRACTICAL USES OF SIMILAR FIGURES

1. All maps, whether of large countries, small sections, cities, railroad charts, or steamship lines, are figures similar to the original.

2. All architects' and engineers' plans, whether they be of a house, a skyscraper, or a railroad bridge, are based on similar figures.

3. All blue prints of the carpenter or cabinet maker show him the ideal he is to fashion and give him a working plan.

4. The designer uses similar figures for costumes, wall paper, carpets, and all kinds of cotton, wool, linen, or silk cloth as well as for all decorative design.

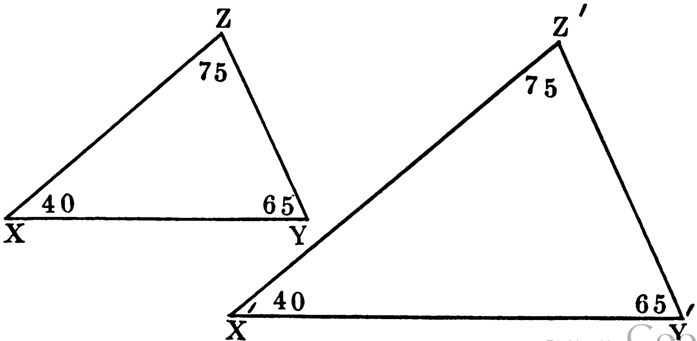
5. On the outside of a paper dress pattern are small outlines of the parts by which they may be identified.

6. The whole process of photography is based on the idea of similar figures.

D. SIMILAR TRIANGLES

Perhaps the most interesting as well as the most important similar figures are similar triangles.

1. (a). Draw two triangles of different sizes whose angles are 40°, 65°, and 75°.



(b) Measure their sides to the nearest tenth of an inch or tenth of a centimeter.

We find

$$\left.\begin{aligned} XY &= 4 \text{ cm.} \\ X'Y' &= 6 \text{ cm.} \end{aligned}\right\} \quad \therefore \frac{XY}{X'Y'} = \frac{4}{6} = .66+$$

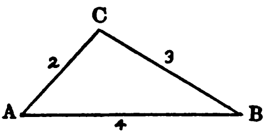
$$\left.\begin{aligned} YZ &= 3.7 \text{ cm.} \\ Y'Z' &= 5.6 \text{ cm.} \end{aligned}\right\} \quad \therefore \frac{YZ}{Y'Z'} = \frac{3.7}{5.6} = .66+$$

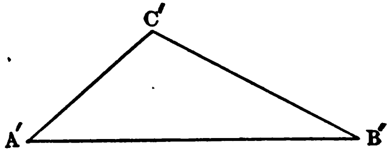
$$\left.\begin{aligned} XZ &= 2.6 \text{ cm.} \\ X'Z' &= 3.9 \text{ cm.} \end{aligned}\right\} \quad \therefore \frac{XZ}{X'Z'} = \frac{2.6}{3.9} = .66+$$

The measures cannot be exact, for all measures are only approximates. But the approximate value of each ratio is .66 +; that is, each side of $\triangle XYZ$ is about $\frac{2}{3}$ of the homologous side of $\triangle X'Y'Z'$.

(c) Are the two triangles similar? Why?

(d) Are two triangles similar if only two angles of one are equal to two homologous angles of the other? What must be true of the third angles?





2. (a) Draw a $\triangle ABC$ whose sides are 2, 3, and 4 cm. long.

(b) Draw $\triangle A'B'C'$ so that the respective sides are each $1\frac{1}{2}$ times as long as the corresponding sides of $\triangle ABC$.

(c) What is the ratio of AB to $A'B'$? of $\frac{BC}{B'C'}$? of $\frac{AC}{A'C'}$?

(d) Compare the homologous ∠s of the two △s.

(e) What kind of △s are ABC and $A'B'C'$?

3. (a) Draw several pairs of triangles in which the corresponding sides have the same ratio.

(b) Measure the homologous ∡ of each pair.

(c) When the corresponding sides of two triangles have the same ratio, what kind of ⊿ are they?

(d) What is true about the homologous angles of such triangles?

4. (a) We say the ratio of 9 to 12 is equal to the ratio of 3 to 4. Written as an equation, $\frac{9}{12} = \frac{3}{4}$.

(b) Such a statement that two ratios are equal is a *proportion.* We say *the numbers are in proportion* or *are proportional.*

(c) In problem 2, we found the ratios of the corresponding sides were equal,

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'}$$

Write two other proportions from these figures.

(d) Fill in the correct words in the blanks:

(1) Two triangles are ———, if their sides are proportional.

(2) Two triangles are ———, if two angles of one are respectively ——— to two angles of the other.

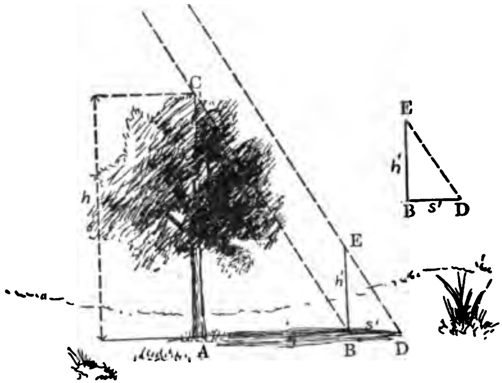
5. (a) *To find the height of a tree from its shadow.*

The tree, the shadow, and the rays of the sun making the shadow may be considered the $\triangle\ ABC$.

(b) Place a stick BE at the end of the shadow of the tree or at any other place in the sunshine. The stick, its shadow, and the sun's rays make the $\triangle\ BDE$.

(c) What is the size of $\angle\ A$ and $\angle\ EBD$?

(d) The sun is so very far away from the earth that its rays are considered parallel.



Compare $\angle ABC$ with $\angle BDE$. Explain.

(e) Is $\triangle ABC \sim \triangle BDE$? Why?

(f) The height of the tree (h) corresponds to the height of the stick (h') and their respective shadows, s and s' correspond.

$$\therefore \frac{h}{h'} = \frac{s}{s'} \quad or \quad \frac{h}{s} = \frac{h'}{s'}$$

Translate these proportions into English statements.

With tape measure the stick and the two shadows.

$$\text{If } h' = 6'$$
$$s = 20'$$
$$\text{and } s' = 4.$$
$$\text{Then } \frac{h}{20} = \frac{6}{4}$$
$$\cancel{20}\left(\frac{h}{\cancel{20}}\right) = \frac{6}{\cancel{4}} \times \overset{5}{\cancel{20}}$$
$$\therefore h = 30$$

The height of the tree = 30 feet.

(g) Thales, a Greek who lived over 600 years B.C., made a trip to Egypt. While there, he astonished

Amasis, the King of Egypt, by finding the height of the pyramid as we found the height of the tree.

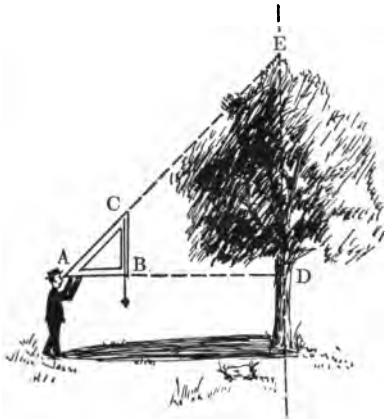
(h) For what achievement did Thales sacrifice an ox to the immortal gods?

(i) Have your teacher tell you or read for yourselves the story of Thales' mule with its load of salt and sponges. Reference, Ball's "Short History of Mathematics."

6. (a) Measure by their shadows the height of your school building, and of the telegraph poles and trees in your vicinity.

(b) If a pole or stick is not convenient, the height of a boy and his shadow may be used.

7. (a) The height of a pole or tree may be found by using a large 45° rt. $\triangle$.



(b) The isosceles $\triangle$ ABC must be held perfectly level. To be sure it is level, a plumb line may be fastened at the corner C. With the $\triangle$ held to

the eye, back away from the tree until the top of the tree is just visible.

(c) Have the distance measured from your eye to the ground and from your toe to the foot of the tree.

$$\triangle ABC \sim \triangle ADE. \quad \text{Why?}$$
$$AD = DE. \quad \text{Why?}$$

(d) What must be added to DE to find the height of the tree?

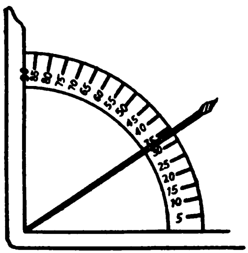
8. Use your triangles to find the heights of various objects in your vicinity.

E. THE USE OF THE QUADRANT AND SEXTANT

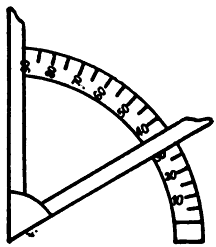
I. Drawings from Bettinus

1. In order to measure angles accurately, a surveyor has a very complicated and very expensive instrument, called a *transit*. In primitive times a much simpler instrument, a *sextant* or a *quadrant*, was used.

2. (a) The instrument was called a quadrant if it was a quarter of a circle, and a sextant if it was a sixth of a circle.



A



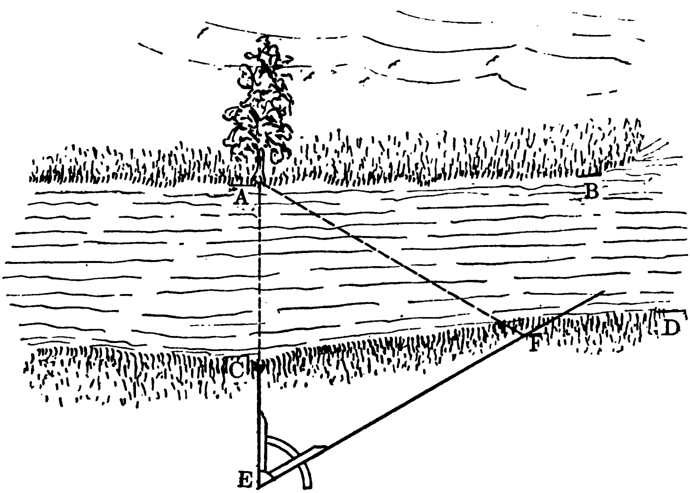
B

(b) An improved and more complex sextant is still used on shipboard for determining latitude.

(c) In its simplest form, that used by Thales, a quadrant is a frame holding a 90° arc, with a moving arm. It is used for measuring angles.

Figures *A* and *B* show two different styles of quadrants.

3. *To find the width of a river, with a tape and quadrant.*



(a) Let *AB* and *CD* be the banks of the river.

(b) Locate some object as *A* on the opposite bank.

(c) Let one pupil stand at *C*, directly opposite this landmark.

(d) Let another pupil walk in the direction of the line *AC* to some point *E*.

(e) With a quadrant, sight an angle of 60°.

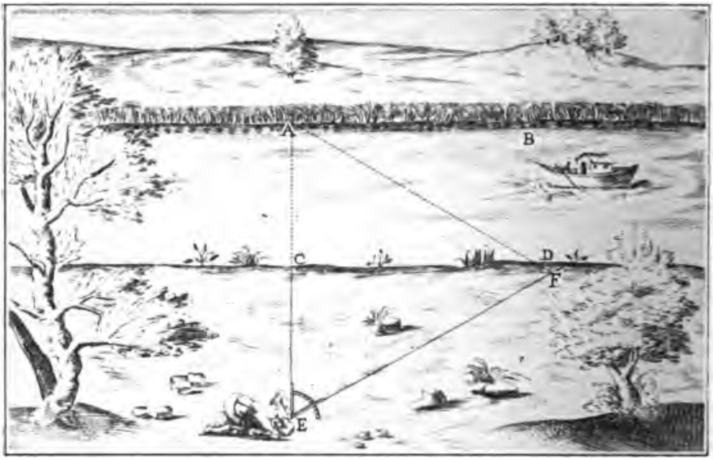
Walk along this line until the point *F* is reached where the $\angle AFE = \angle E$.

(f) What kind of a △ is *AFE?*

(g) What lines are equal?

(h) Measure *EF* and *EC* to find *AC*. How?

4. (a) The drawings shown are taken from an old book by Bettinus, printed early in the seventeenth century. They illustrate the early use of the quadrant.



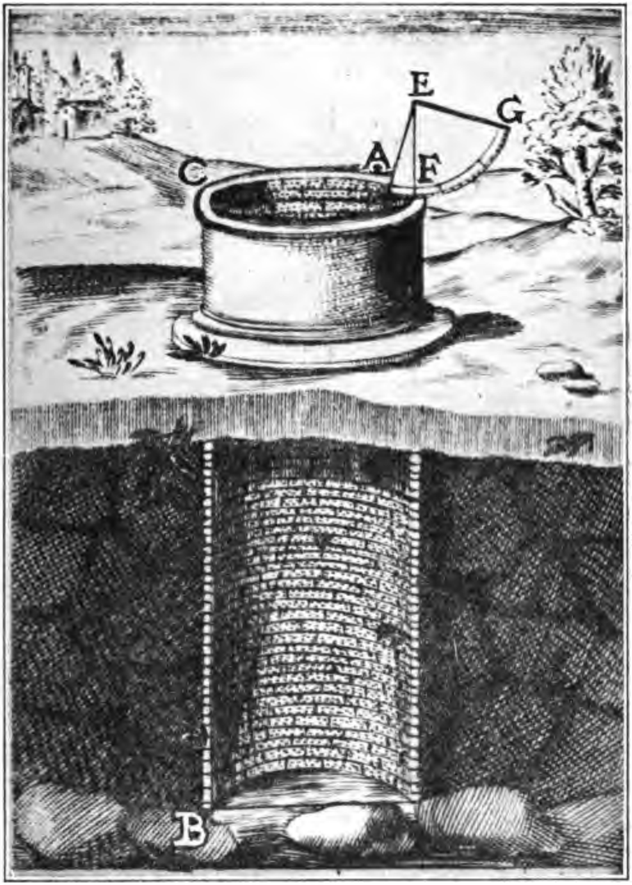
(b) The drawing from Bettinus on page 153 shows the use of a quadrant in finding the depth of a well.

Which line measures the unknown depth?

(c) Find two similar triangles in the figure.

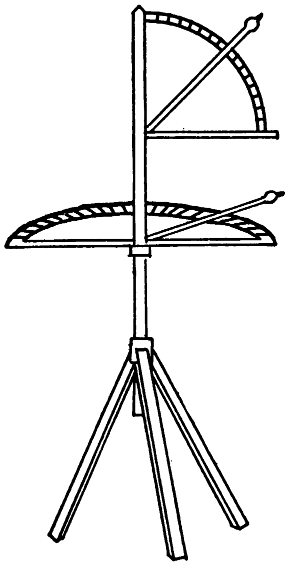
(d) Measure three lines and use in a proportion to find the depth of the well.

(e) Assume reasonable values for these lines and compute the depth.

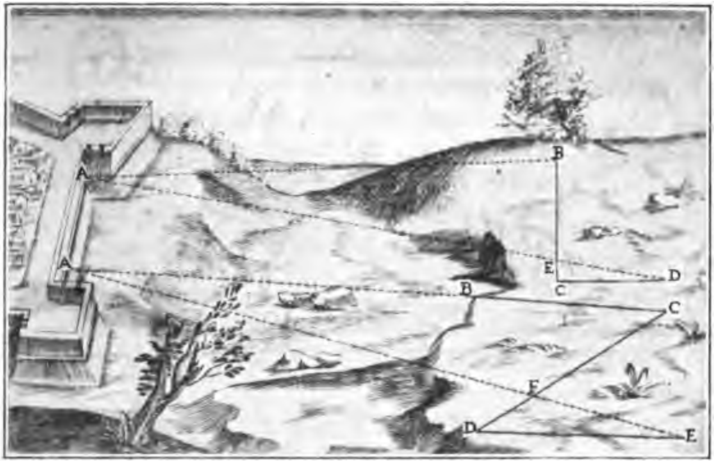


5. By mounting a quadrant or protractor on a frame a very good substitute for a transit may be made. This will be more serviceable if one is mounted in a horizontal position and another in a vertical position.

Any pupil who is clever with his hands can make such a substitute for a transit.



6. (a) Suppose A is a hostile camp, a ship at sea, an island, or other inaccessible spot.
   (b) A gunner at B wants to know the distance AB. He makes out a line $BC \perp AB$, and a line $CD \perp BC$. These lines may be any length.
   (c) From D, he sights to A and notes the intersection with BC at E.
   (d) By measuring BE, EC, and CD, he can find the distance AB. How and why?
   (e) Find AB if $BE = 24'$
   $EC = 3'$
   and $CD = 40'$.
7. (a) The distance may be found in another way.



(b) Suppose AB is the unknown distance in the lower figure.
(c) Prolong AB to any point C.
(d) Draw any line from C as CE.
(e) Draw $DE \parallel BC$.
(This may be done by making $\angle\ CDE = \angle\ C$.)
(f) Sight from E to A. Mark the intersection F.
(g) What $\triangle$ are similar? Why?
(h) Measure BC, CF, FD, and DE. Find AC and then AB.

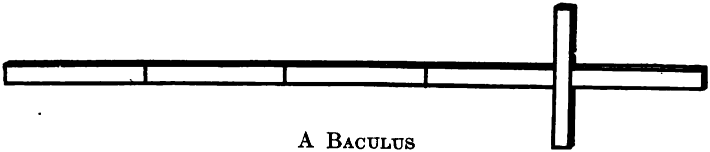
(i) Note: This problem is taken from a book written in Latin and printed in 1645. It is the same book from which the other drawings were taken.

8. By using one of these methods, find the width of a river, creek, or street in your vicinity.

9. (a) In the measurements thus far, only one end of the line has been inaccessible. There is a very easy method for measuring distances, both ends of

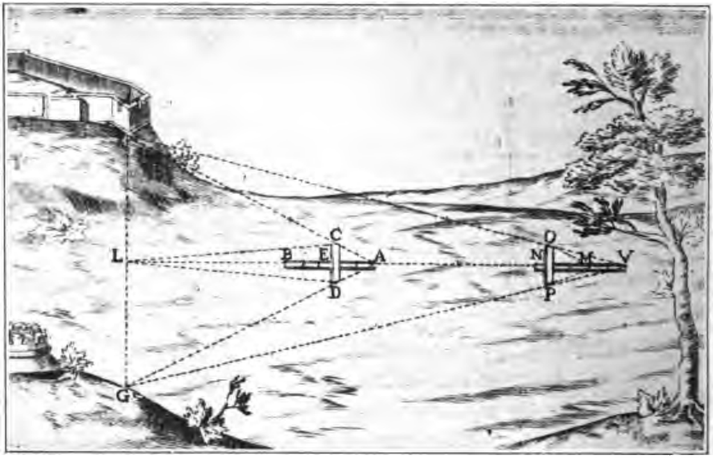
which are inaccessible. It is by means of an old instrument called a *baculus*, meaning *rod.*

(b) The baculus is really made of two rods, one very much longer than the other and marked off in segments of equal length. The shorter rod is equal in length to one of the segments and is made to slide over the longer one easily, but always remains perpendicular to it.



A BACULUS

10. The following picture and explanation showing the use of the baculus are also taken from the book by Bettinus.



(a) FG is the distance required.

(b) Let CD be at a certain mark on the baculus AB.

(c) Sight from A so that points F and G are just seen along C and D.

(d) Then move the shorter rod nearer A, if you approach FG; or nearer B, if you move away from FG for the next observation.

(e) Find the point V, so that F and G may still be just visible past the ends of the shorter rod at O and P.

(f) By measuring the distance AV, between the two stations, the desired length FG will be had.

NOTE: The above explanation is translated from the Latin. The proof is by proportions derived from several sets of similar triangles, but it is too difficult to be given here. The construction and use of the baculus, however, are very simple.

11. Make estimates of distances between objects on the opposite bank of a river, then measure with a baculus and tape.

12. *Another way to measure inaccessible distances.*

Let AB be the required distance. Standing at C let observer sight A through D on a rod placed at B.

Do likewise at F, some other convenient point.

Draw BC, CF, and EF.

Draw $BH \parallel EF$.

It is proved in geometry that if a line is parallel to one side of a triangle, it divides the other two sides proportionally.

Therefore, $$\frac{CH}{HF} = \frac{BC}{AB}.$$

Which three lines can be measured to find AB?

13. ***To measure an inaccessible distance by a quadrant with a plumb line, by drawing a small similar triangle.***

Let AB be the required distance.

What kind of a triangle is ABC?

With the quadrant measure angle DCE.

Draw a small right triangle with $\angle F = \angle C$.

Then $$\frac{FG}{GH} = \frac{AC}{AB}.$$

Which three lines can be measured to find the distance AB?

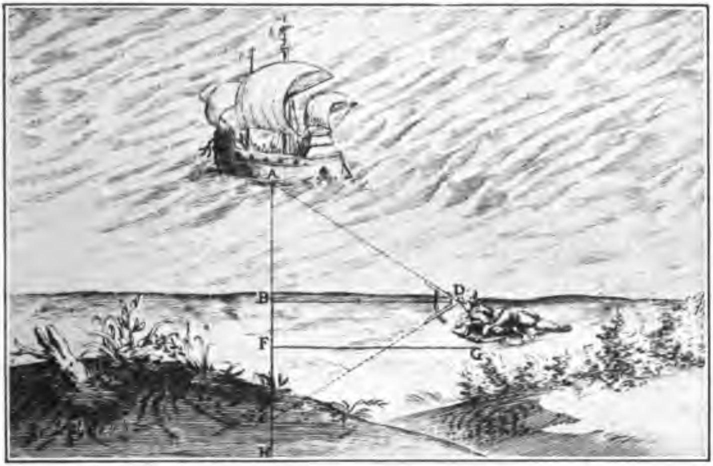
The name of this book by Marius Bettinus is,

Apiaria Universae Philosophicae
Mathematicae
Progymnasma Primum

The first problem is called Proposition I. An exact translation of it is given below.

"PROPOSITION I. *To measure an inaccessible distance by the twenty-sixth proposition of Book I of Euclid.*

"Method used by Thales to measure distances of ships at sea.



"Let *A* be the position of a ship at sea and let Thales be on the shore at *B*. How shall he find the distance *AB?*

"Let him withdraw in a straight line *AB* to any desired point, as *C*. At *B*, with the aid of a norma, make a perpendicular and mark off any length, as *BD*.

"Then let the angle *BDA* be noted. On the other side, let the angle *BDC* be marked off equal to the angle *BDA*.

"I assert (Thales says) that if you measure the distance *BC*, you will know the desired distance *AB*.

"Scholium to Proposition I.

"*If the point B is on uneven ground, measure off any distance BF, draw FG perpendicular to AC and operate from G.*

"*Subtract BF from H to get point C.*"

NOTE: A norma is another instrument used by the ancients to draw perpendicular lines. It consists of three rods placed at right angles to each other.

Note the lack of perspective in the drawing.

If the angles *BDA* and *BDC* are equal, what kind of a triangle is *ADC?*

Why does *BC* equal *AD?*

Bettinus gave a proof for this proposition by proving the triangles *ABD* and *CBD* equal.

CHAPTER ELEVEN

POLYGONS

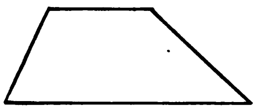
A. TRAPEZOIDS

1. (a) Roads and streets do not always run at right angles to each other, but lot lines are usually perpendicular to the street on which the lots face. In such a city block most of the lots are rectangular, but a few will be in the shape of triangles and trapezoids.

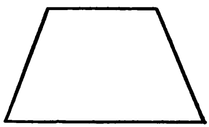
(b) A *trapezoid* is a figure inclosed by four straight lines, only two of which are parallel.

(c) The two parallel sides are the bases. The two non-parallel sides are the legs.

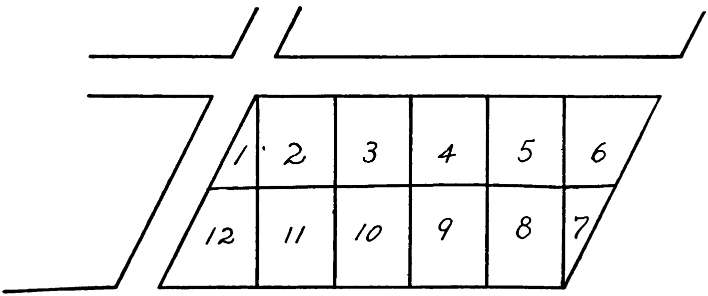
(d) If the two legs are equal, the figure is an isosceles trapezoid.



TRAPEZOID



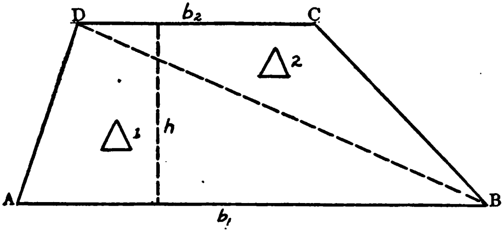
ISOSCELES TRAPEZOID



2. (a) What is the shape of each lot in the above plot?

(b) In order to find the area of lots 6 and 12, one must know how to find the area of a trapezoid.

(c) The symbol for trapezoid is ⏢.



3. (a) Draw a trapezoid $ABCD$ and its altitude h.

(b) Draw one diagonal as BD.

(c) Into what two parts does BD divide the trapezoid?

(d) We see that $\triangle\ 1 + \triangle\ 2 =$ ⏢ $ABCD$.

(e) Let b_1 (read b sub 1) = the base of $\triangle$ 1 or the lower base of the ⏢; and b_2 (read b sub 2) = the base of $\triangle$ 2 or the upper base of the ⏢.

(f) What is the height of each $\triangle$? The height of the trapezoid is a line perpendicular to the bases.

(1) $S_{\triangle 1} = \frac{1}{2}\, b_1 h$

(2) $S_{\triangle 2} = \frac{1}{2}\, b_2 h$

(3) $S_{⏢} = \frac{1}{2}\, b_1 h + \frac{1}{2}\, b_2 h$

(4) Take out the common factor $\frac{1}{2}\, h$.

(5) $\therefore\ S_{⏢} = \frac{1}{2}\, h\ (b_1 + b_2)$.

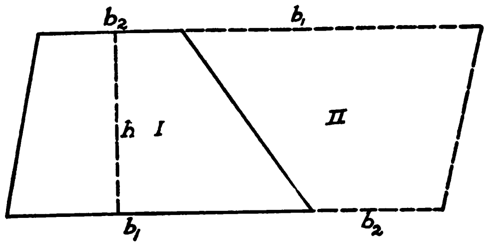
(g) Translate this formula into an English statement.

4. If $b_1 = 10$ cm., $b_2 = 5$ cm., and $h = 3$ cm., find $S_{⏢}$.

Solution.

$$\text{Then } S_{⏢} = \tfrac{1}{2}\, h\ (b_1 + b_2)$$
$$= \tfrac{1}{2} \times 3\ (10 + 5)$$
$$= \tfrac{1}{2} \times 3 \times 15$$
$$= \tfrac{45}{2}$$
$$= 22\tfrac{1}{2} \text{ sq. cm.}$$

5. The formula for the area of a trapezoid may be found in a different way.



(a) Draw two trapezoids that are exactly equal.
(b) Cut out one and place it beside the other as in the illustration.
(c) What kind of a figure is the result?
(d) What is the base of the new figure? The height?
(e) What is its area?
(f) What part of the new figure is the trapezoid?
(g) Therefore, what is the area of the trapezoid?

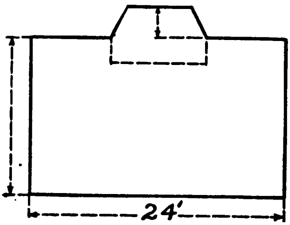
6. (a) Find the areas of the following trapezoids.
(b) Use the formula

$$S_{\text{trap}} = \tfrac{1}{2}\, h\, (b_1 + b_2).$$

No.	b_1	b_2	h	S
(1)	10″	5″	4″	......
(2)	8″	6″	7″	......
(3)	2′ 6″	1′ 6″	9″	......
(4)	140′	50′	96′	......
(5)	$13\frac{1}{2}'$	$8\frac{1}{2}'$	$5\frac{1}{2}'$	......
(6)	$a + 3$	$a + 2$	$4a$	......
(7)	$2x + 7$	$2x + 3$	$3x$	......
(8)	$a + 2b$	$a + b$	$2a$	......
(9)	$2c + 3d$	$c + 2d$	$2c$	......
(10)	18.3′	10.5′	8.2′	......
(11)	$x + y$	x	$2x$	......
(12)	165.7′	95.1′	120′	......

7. In the diagram page 161 measure lots 1, 6, 7, 8, and 12 to the nearest tenth of a centimeter. Find their areas if 1 cm. equals 60 ft.

8. (a) A room has a bay window and is shaped like the diagram.



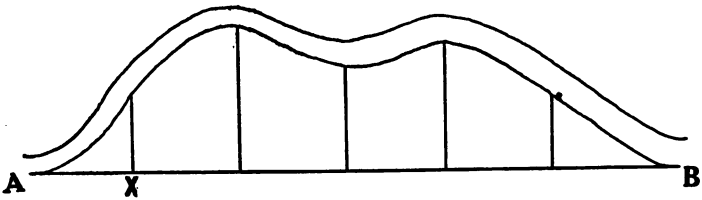
(b) The length of the room is 24 feet.

(c) Find the scale and the entire area of the room including the bay window.

9. (a) Draw a trapezoid with a very small upper base. What other figure does it resemble?

(b) Show that the formula for the area of a trapezoid may be used for the triangle by considering the upper base 0.

10. (a) To measure an irregular shaped piece of land, as that in the bend of a river, a straight line may be run so as to cut off the bend, as *AB*. At equal intervals on *AB* perpendiculars are run. These divide the land into approximate trapezoids.



(b) This figure is drawn to a scale of 1″ to 40′. Find the length of parts *AB* and of the perpendiculars to *AB*.

(c) Find the area of each near trapezoid.

(d) Find the total area of the strip of land in the bend.

(e) The perpendicular lines are called offsets.
Find the sum of the offsets and multiply it by common altitude AX. How does this result compare with your total area? Show why this method may be used if the offsets at the ends A and B are 0?

B. OTHER POLYGONS

1. Any figure bounded by three or more straight lines is a polygon.

The word polygon means *many sided.*

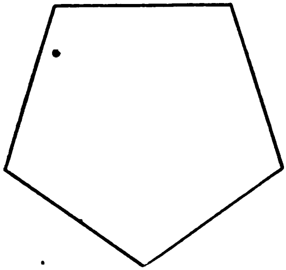
What special kinds of polygons have you used?

2. Polygons have special names according to the numbers of their sides.

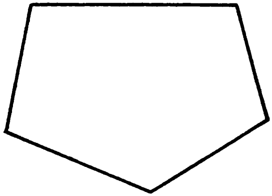
(a) A triangle is a polygon with 3 sides.
(b) A quadrilateral is a polygon with 4 sides.
(c) A pentagon is a polygon with 5 sides.
(d) A hexagon is a polygon with 6 sides.
(e) An octagon is a polygon with 8 sides.
(f) Other polygons have special names, but these are the most common. They are used extensively in design, especially for tile, linoleum, and wall paper patterns.

3. (a) If all the sides of a polygon are equal and all of its angles are equal, it is a *regular* polygon.
(b) What kind of a triangle is regular?
(c) What kind of a quadrilateral is regular?
(d) Is a rhombus a regular polygon? Why?
(e) Name some special kinds of triangles.
(f) The word polygon alone means an irregular one. A regular polygon is specially mentioned.

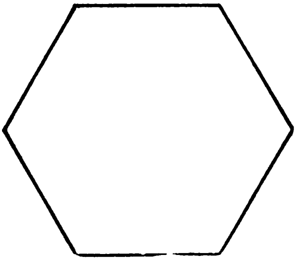
(g) The following figures show the regular and irregular polygons of five, six, and eight sides.



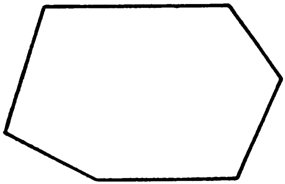
REGULAR PENTAGON



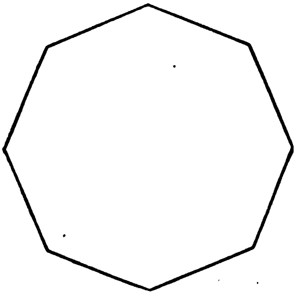
PENTAGON



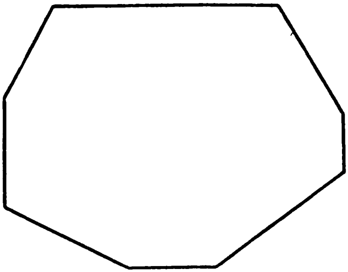
REGULAR HEXAGON



HEXAGON



REGULAR OCTAGON



OCTAGON

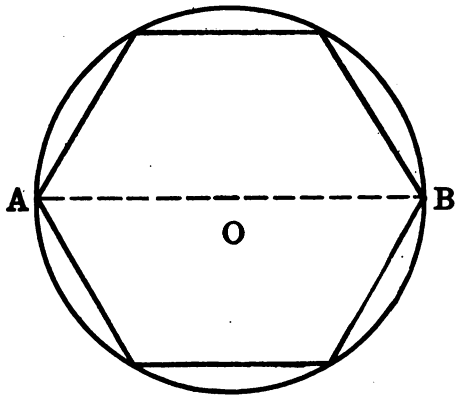
C. CONSTRUCTION OF REGULAR POLYGONS

A polygon is *inscribed* in a circle when it is drawn inside the circle so that each vertex lies on the circumference.

The word *inscribe* literally means *to write in* or *to draw in*. The regular hexagon is the easiest to construct, therefore we shall learn that first.

I. The Hexagon

1. *To construct a regular hexagon.*



(a) Draw a circle with the desired radius.

(b) Draw any diameter, as AB.

(c) With A and B as centers and with the original radius draw arcs cutting the circle on each side of A and B.

(d) Join the points of intersection in succession to make a regular hexagon.

(e) NOTE: In making constructions, do not lift the compass point until all possible arcs from that center are made.

2. (a) Make another regular hexagon of a different size.

(b) Measure the ∠ at each vertex in both figures.

(c) Draw all the radii. What shaped figures result? Measure their sides.

(d) How many degrees are in each ∠ of each △?

3. (a) Make a third regular hexagon.

(b) Join three alternate vertices with straight lines.

(c) What kind of a figure results?

4. (a) Draw another regular hexagon, a rather large one with a radius of about $1\frac{1}{2}''$.

(b) Draw all the radii.

(c) Measure each ∠ at the center.

(d) Bisect each ∠ at the center by other radii.

(e) Join each of these new points on the circumference to the two nearest vertices of the hexagon.

(f) How many sides has the new figure? Are they equal?

(g) In Latin *duo* means *two*, *decem* means *ten*, and *duodecim* means *two plus ten* or *twelve*. Similarly, in Greek the word for *twelve* is *dodeka* and for *angle* is *gonia*. Therefore this new twelve-sided polygon is called *dodecagon*.

5. (a) Draw a regular hexagon and an irregular one.

(b) Measure the angles at the vertices in each.

(c) Find the sum of these angles in each hexagon.

(d) Change the measure of these sums in degrees to a measure in right ∡.

How do the two sums compare?

6. (a) How many regular hexagonal tiles will exactly fit together at one point?

(b) How many square tiles?

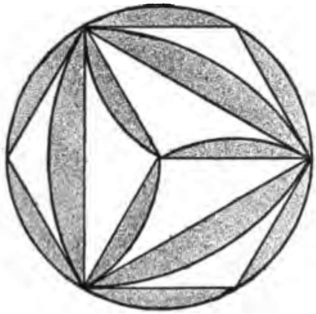
(c) How many regular triangular tiles?

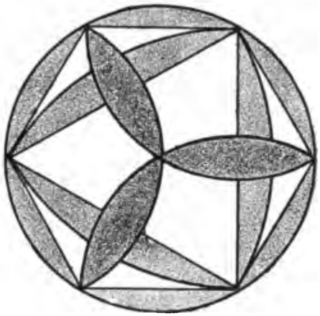
7. Draw a regular hexagon whose side is $2''$; $1\frac{3}{4}''$.

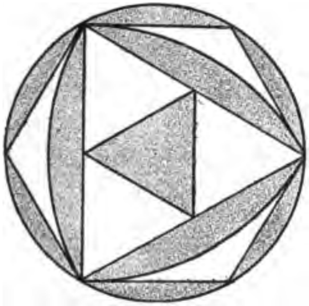
8. (a) Draw a regular hexagon and three radii to alternate vertices.

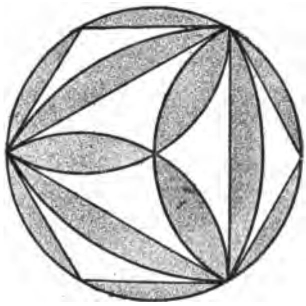
(b) What shaped figures result?

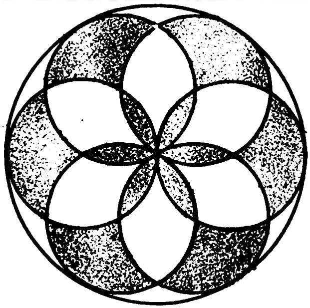
(c) Join the ends of these radii.
(d) How do the resulting triangles compare?
(e) How many of these triangles are in the hexagon?











(f) How many are in the inscribed triangle?
(g) What is the ratio of the triangle to the hexagon?

9. (a) With each vertex as a center and with the original radius draw arcs within the hexagon to make a conventional flower.

(b) Draw these only at alternate vertices.

10. By combining different lines and arcs within a regular hexagon, and shading some of the parts, different decorative designs may be made.

11. (a) Draw a regular hexagon in a circle.

(b) Join three alternate vertices to make a regular triangle.

(c) With each vertex of the triangle as a center and its side as a radius, draw an arc.

(d) Shade or color different parts.

12. (a) Draw a regular hexagon.

(b) Bisect each side. (Bisect one side and use its half as a measure.)

(c) Join the mid-points of the sides in succession.

(d) What kind of a figure is formed?

13. (a) Draw a regular hexagon.

(b) Join the six pairs of alternate vertices.

(c) How many points are in the resulting star?

(d) What is the shape of the figure left in the center?

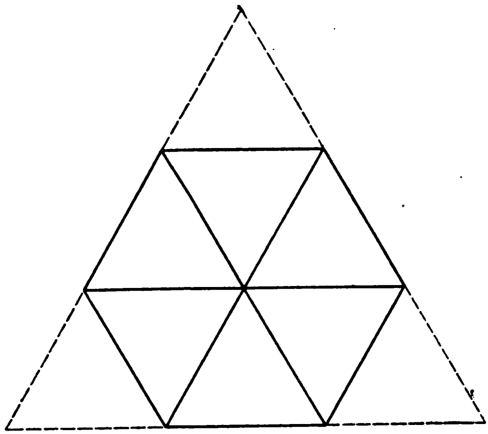
14. *To make a regular hexagon from an equilateral triangle.*

(a) Draw and cut out an equilateral triangle about 10 cm. on a side.

(b) Either by folding or drawing two altitudes, find the center of the triangle, that is, the intersection of the altitudes.

(c) Fold over each corner of the triangle, so that the vertex just touches the center.

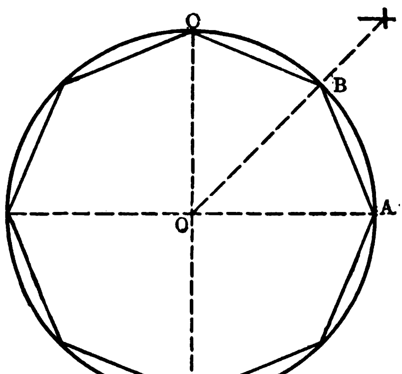
(d) Measure the sides of the resulting hexagon.



II. The Octagon

The construction of the octagon is based upon the square.

1. *To construct a regular octagon.*



(a) Draw a circle with two ⊥ diameters.

(b) Bisect each angle at the center O.

Only one ∠ need be bisected, as $\angle AOC$, for an ⌒ equal to AB may be laid off on each quarter.

(c) Join the eight points on the circumference in succession to make a regular octagon.

2. (a) Make another regular octagon of different size.
   (b) Measure the ∠ at each vertex in both figures. How many degrees in each? How many right ∠s?
   (c) Draw all the radii.
   (d) How many and what kind of △s are formed?
   (e) How many degrees are in each ∠ of each △?

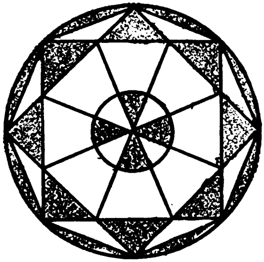
3. What would you do to an octagon to make a 16-sided polygon?

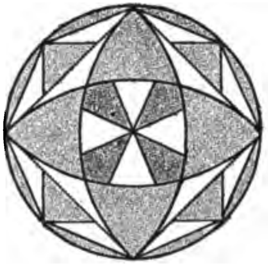
4. (a) Draw a regular and an irregular octagon.
   (b) Measure the angles at the vertices in each.
   (c) Find the sum of these angles in each octagon.
   (d) What does the sum measured in degrees equal when measured in right angles?
   (e) How do the two sums compare?
   (f) Does the sum of the interior angles of an octagon change with its size or shape?

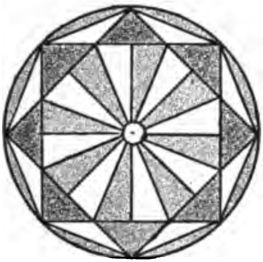
5. (a) Can you put three regular octagonal tiles together so that the corners exactly fit?
   (b) If you put two together, will there be any space left?
   (c) What shaped figure will exactly fit in the space left?

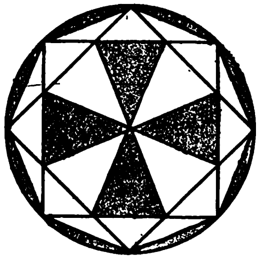
6. (a) Join the four alternate vertices of a regular octagon in succession.
   (b) What shaped figure is formed?
   (c) Join the other four pairs of alternate vertices.
   (d) What shaped figure is left in the center?

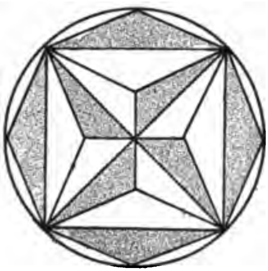
7. (a) Draw several octagons in circles.
   (b) By joining alternate vertices, drawing diameters and arcs of circles and shading or coloring different parts, make different designs.

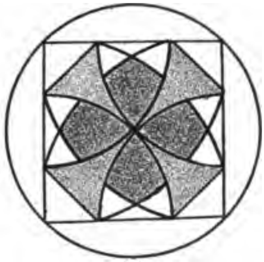


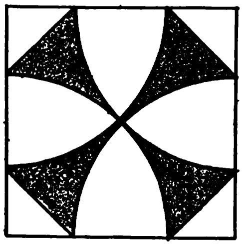












8. *To make a regular octagon from a square by folding.*

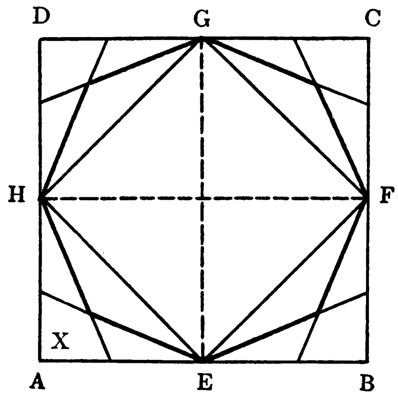
(a) Draw and cut out a square 10 cm. long.

(b) By folding bisect each side of the square.

(c) Fold over the corners, making an inner square *EFGH*.

(d) Open the paper and then fold over the corner *A* so that *AE* falls along *EH*.

(e) Open again and fold over the same corner so that *AH* falls along *EH*.



(f) These folds will intersect, as at *X*.

(g) In like manner fold over each corner.

(h) The resulting figure is a regular octagon.

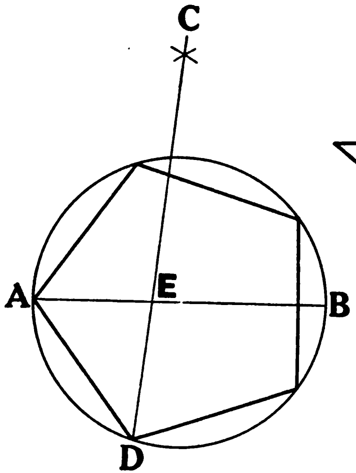
III. The Pentagon

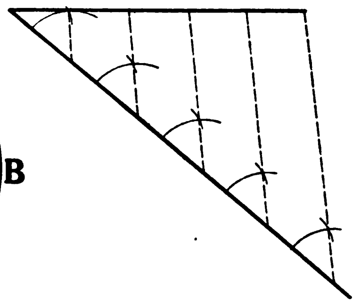
1. *To construct a regular pentagon.*

The greatest care must be used and pencils must be very sharp in the construction of a pentagon. If the compass point is off by the width of a pencil line, the pentagon will not be exact.

(a) Construct a circle and divide the diameter *AB* into five equal parts.

(b) Find *C*, the vertex of an equilateral △ on *AB*. (The △ need not be drawn.)





(c) Through E, the second mark on the diameter, draw a line from C to meet the circumference at D.

(d) Draw AD.

(e) Use AD as a radius and mark off four other equal arcs on the circumference.

(f) Join these points in succession to form a regular pentagon.

2. (a) Make another regular pentagon.

(b) Measure the ∠ at each vertex in both figures. How many degrees are in each ∠? How many right ⦣?

(c) Draw all the radii.

(d) How many and what kind of ⚠ are formed?

(e) How many degrees are in each ∠ of each △?

3. What would you do to the pentagon to make a regular 10-sided polygon, that is, a decagon?

4. (a) Draw an irregular pentagon.

(b) Measure the ⦣ at the vertices and find their sum, measured in degrees and in right ⦣.

(c) How does the sum of the $\angle$s in this one compare with the sum of the $\angle$s of a regular pentagon?

(d) Does the sum of the interior $\angle$s of a pentagon change with its shape or size?

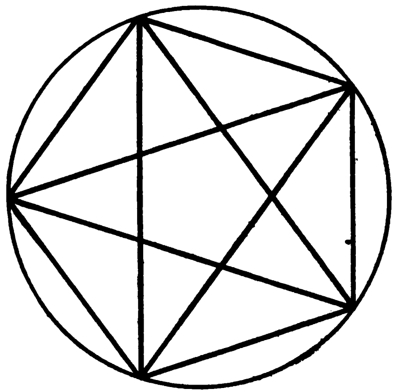
5. Does the sum of the interior $\angle$s of a polygon change with the number of sides?

6. Make a table showing the sum of the interior $\angle$s of the different polygons constructed and the size of the $\angle$ at each vertex, measured in rt. $\angle$s and in degrees.

Angles of Regular Polygons

Name of Polygon	Number of Sides	Sum of the Interior $\angle$s	No. of Rt. $\angle$s in each $\angle$	No. of Degrees in each $\angle$
Triangle.....	3	2 rt. $\angle$s	$\frac{2}{3}$ rt. $\angle$	60°
Square......	4			
Pentagon....	5			
Hexagon.....	6			
Octagon.....	8			

7. (a) Draw a regular pentagon and all possible diagonals.



(b) How many points are in the star?

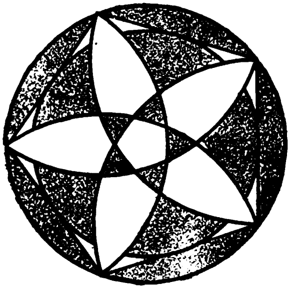
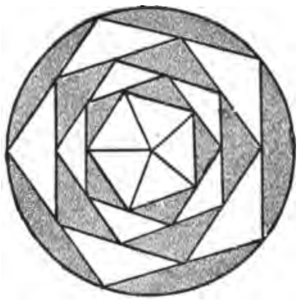
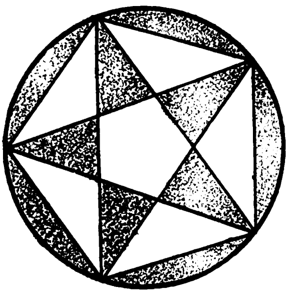
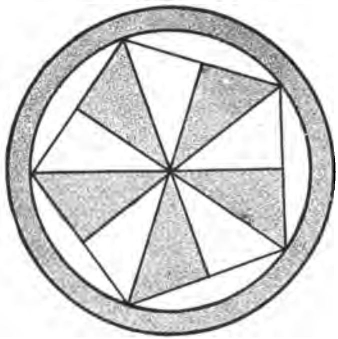
(c) What kind of $\triangle$s are formed on each side of the pentagon?

(d) What kind of a polygon is left in the center?
(e) Compare its position with the original pentagon.
(f) There are now three $\angle$s at each vertex.
Compare them in size.
(g) How many isosceles $\triangle$s can you find in the figure?
(h) Without measuring, can you compute the size of each of the four $\angle$s at one vertex of the inner pentagon?

8. Can you find a rhombus in the figure? How many?

9. (a) Can you fit four regular pentagonal tiles together?
(b) Can you fit three together?
(c) How much of an angle will be left?
(d) Is there any regular polygon that will fit in exactly?

10. Make and shade or color several original designs made from pentagons.

11. It is impossible to construct exact regular polygons of 7, 9, or 11 sides with compass and ruler. But the method given for the pentagon, which gives approximate results only, may be used for the other polygons.

To construct a heptagon or 7-sided polygon, divide the diameter into 7 equal parts, and from the vertex of the equilateral triangle draw a line through the second point of division. Proceed as in the construction of the pentagon.

CHAPTER TWELVE

CYLINDERS AND CIRCLES

A. CYLINDERS

1. We have seen that many objects in our surroundings are more or less rectangular. But there is another group of objects about us that have one general shape different from the rectangular solid.

2. Think for a moment of the shape of a telegraph pole, the trunk of a tree, the water glass for the table, the ice cream freezer, the fruit jar, the rows of tin cans containing vegetables on the grocer's shelves, smoke stacks on steamships, the pencil with which you write, the pillars in front of some public buildings. All of these things are cylindrical in shape, or like a cylinder.

3. The tin can is probably the most convenient example of a cylinder.

(a) How many surfaces has it?

(b) How many of these are flat or plane?

(c) How many are curved?

(d) What is the shape of the flat surfaces?

(e) What is their relative positions?

4. (a) To test a plane surface, put the edge of your ruler on it in several different positions. If the edge touches the surface in all positions, it is a flat or plane surface.

(b) Put the edge of your ruler along the side of a tin can or other cylindrical object, from top to

bottom. Does the edge touch the surface at all points?

(c) Put the edge in several other positions and see if it touches.

(d) If the edge does not touch the surface in all positions, the surface is curved.

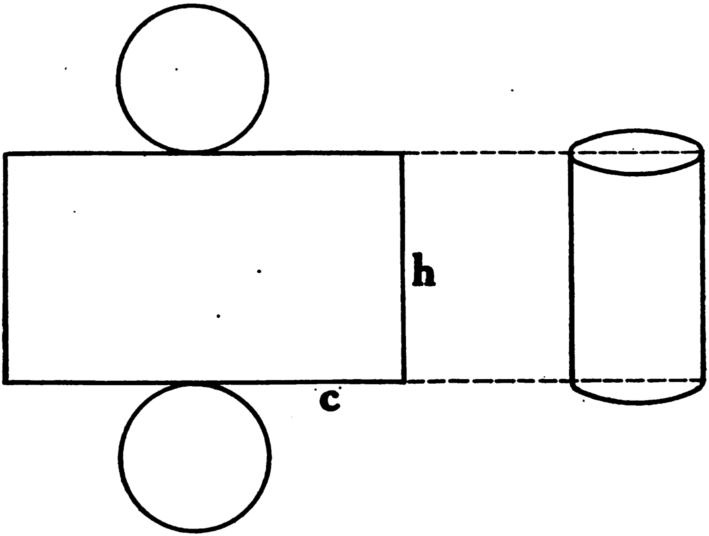
(e) There are many kinds of curved surfaces.

(1) A ball has one kind, called a *spherical surface.*

(2) The side of the tin can is another, called a *cylindrical surface.*

(f) Revolve a rectangular card about one edge. The opposite edge traces a *cylindrical surface* and the whole card traces a *cylinder*.

5. The only kind of a cylinder we shall consider is one in which the side is perpendicular to the base.



6. A *cylinder* is a figure enclosed by a cylindrical surface and two parallel circular bases.

7. (a) Imagine a tin can cut straight down the side and the top and bottom almost cut off and the tin pressed out flat.

An illustration of this pattern is shown on page 180.

(b) The cylindrical surface is called the lateral surface of the cylinder. What is the shape of the lateral surface in the pattern?

(c) What is the height of this rectangle?
What is its base?

(d) We see, therefore, in order to find out how much tin it will take to make this can, or how big to make the label for it, we must know how to find the circumference of a circle.

B. CIRCLES

I. Circumference of a Circle

1. (a) Carefully measure around the can and measure its diameter.

(b) How many times larger than the diameter is the circumference?

(c) Measure a glass and several other cylinders in the same way and find the ratio of the circumference to the diameter in each case.

(d) Is the ratio the same for all circles?

(e) We cannot measure this ratio exactly, but it is nearly $3\frac{1}{7}$. That is, every circumference is about $3\frac{1}{7}$ times its diameter, or, as a decimal, 3.1416 times.

(f) This ratio is used a great deal by mathematicians and for convenience they use as a symbol for it the Greek letter for *p*, called *Pi*. The symbol is π.

(g) The circumference of a circle is $3\frac{1}{7}$ times its diameter.

Let c = circumference
$\pi = 3\frac{1}{7}$ or 3.1416
and d = diameter.

Then the formula is

$$c = \pi d.$$

(h) Since the diameter is two times the radius, we may put $2r$ in place of d, and have

$$c = 2\pi r.$$

Translate this formula into an English statement.

2. For the cylinder

$$\text{Lat. } S = ch = 2\pi rh \text{ or } \pi hd.$$

Translate into an English statement.

3. A coffee can has a $4\frac{1}{2}''$ diameter and is $5\frac{1}{4}''$ high. How much paper will be needed to furnish labels for 5000 cans?

4. (a) A farmer's silo is 12 feet in diameter and 40 feet high. How many square feet are in its lateral area?

(b) How many gallons of paint are needed to paint the outside, allowing one gallon to 250 sq. ft.?

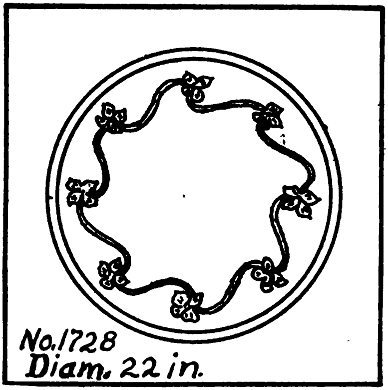
(c) If a painter can paint one square an hour and receives \$.65 per hour, how much does he receive for the painting?

(d) What is the total cost to the farmer?

5. A girl is buying a stamped doily to embroider. At the same time she wants to buy the lace edge for it. The circular doily is stamped on a square of linen. In one corner is stamped the diameter of the doily.

(a) How much lace 1″ wide will be needed for a 22″ doily?

(1) What will be the diameter of the finished doily after the lace is sewed on?



(2) In order to have the lace lie flat, the exact amount plus ½″ to ¾″ for the seam is needed.

(3) No allowance need be made for "fulling on," because the circumference of the doily itself is enough smaller than the edge of the lace to allow for fulling.

(b) What will the lace cost at $.18 a yd.?

6. A circular lunch cloth just covers the top of a 54″ round table.

(a) How much lace 3½″ wide will be needed for the edge of the cloth?

(b) What will the lace cost @ $.37½ a yard?

7. A luncheon set contains one 24″ centerpiece, four 7″ plate doilies, and eight 3″ tumbler doilies.

The centerpiece has lace 1½″ wide @ $.35 a yard; the plate doilies have lace ¾″ wide @ $.25 a yard; and the tumbler doilies have lace ½″ wide at $.18 a yard.

(a) How much lace of each kind is needed?

(b) Find the cost of each kind and the total cost of the lace.

8. (a) The wheels of one automobile have 32″ tires. How many revolutions will the wheels have to make in going one mile?

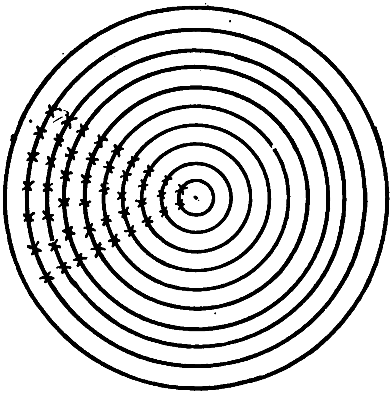
(b) The wheels of another machine have 36″ tires. Which wheels make more revolutions per mile, and how many more?

(c) Such calculations have to be made in setting speedometers, for a speedometer set for one sized tire will not register correctly if a different sized tire is put on the machine.

(d) Find the circumference and the number of revolutions per mile for a 33″ tire; for a 38″ tire.

9. A circular flower bed is ten feet in diameter.

(a) How much wire fencing is needed for it?



(b) Two or more circles are concentric if they are drawn about the same center with different radii.

Con-centric means *centers together.*

(c) The owner of the flower bed wants to fill it with aster plants, putting one at the center and the others in concentric circles 6″ apart. In each circle the plants are 6″ apart. The outermost circle is within 6″ of the fence.

(d) How many plants must be purchased?

II. Area of Circle

We have found how to measure the lateral area of a cylinder. But in order to find how much tin it will take to make the can we must know how to find the area of the top and bottom; i.e., to find the area of a circle.

1. *To find the area of a circle.*

(a) Draw and cut out a circle.

(b) Fold over on its diameter.

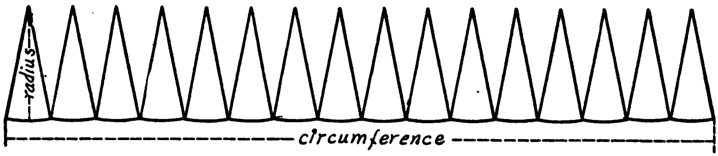
(c) Fold again and again until it looks like a very small piece of pie.

(d) Unfold and count the number of parts. There should be at least sixteen.

(e) Cut through the circumference to the center in just one place.

(f) Cut from the center along each fold *almost* to the circumference.

(g) The circle has been cut into a series of parts that are very much like triangles.



(h) The height of each triangle is the radius of the circle and the sum of the bases of the $\triangle$ is the circumference of the circle, or $2\pi r$.

$$S \text{ of } \triangle = \frac{1}{2} bh$$

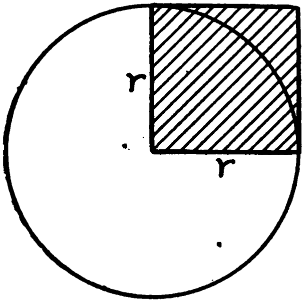
Substituting $2\pi r$ for b, and r for h, we get

$$S_{\odot} = \frac{1}{2} \times 2\pi r \times r$$

$$S_{\odot} = \pi r^2.$$

(i) Translate this formula for the area of a circle into an English statement.

2. (a) Draw a circle.
   (b) Draw a square on one radius.



   (c) What is the area of this square?
   (d) How many times larger than the square is the circle?
   (e) The formula $S_{\odot} = \pi r^2$ means that the circle is about $3\frac{1}{7}$ times as large as a square drawn on the radius.

3. Find the area of a circle 7″ in diameter.

Solution.

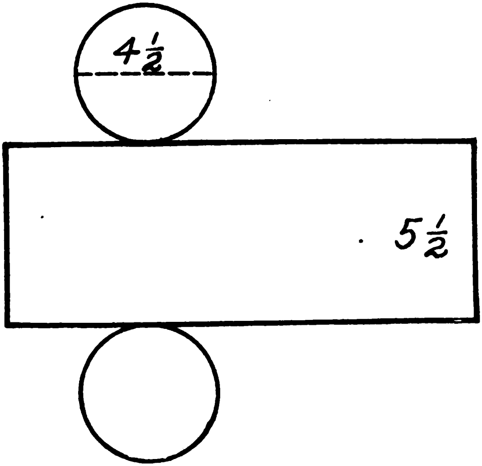
$$\text{Formula } S_{\odot} = \pi r^2$$
$$\text{Given } d = 7''$$
$$\text{Then } r = \frac{7}{2}$$
$$r^2 = \frac{7}{2} \times \frac{7}{2}$$
$$S_{\odot} = 3\frac{1}{7} \times \frac{7}{2} \times \frac{7}{2}$$
$$= \frac{\overset{11}{\cancel{22}}}{\cancel{7}} \times \frac{7}{\cancel{2}} \times \frac{7}{2}$$
$$= \frac{77}{2} = 38\tfrac{1}{2} \text{ sq. in.}$$

(a) Instead of squaring $\frac{7}{2}$ and using $\frac{49}{4}$, the square was indicated as $\frac{7}{2} \times \frac{7}{2}$. This plan often saves much computation. In this problem two cancellations could be made.

(b) It is better *not* to perform a multiplication or division until it becomes necessary.

C. SURFACE OF A CYLINDER

1. Find the number of square inches of tin in a can $4\frac{1}{2}''$ in diameter and $5\frac{1}{2}''$ high.



Solution.

$$\text{(a) Given } d = 4\tfrac{1}{2} \text{ or } \frac{9}{2}$$

$$r = \frac{9}{4}$$

$$r^2 = \frac{9}{4} \times \frac{9}{4}$$

$$2 \text{ bases} = 2 \times \pi r^2$$

$$= \cancel{2} \times \frac{\overset{11}{\cancel{22}}}{7} \times \frac{9}{\underset{2}{\cancel{4}}} \times \frac{9}{4}$$

$$= \frac{891}{28}$$

$$= 31.8 \text{ sq. in.}$$

$$\text{(b) Given } r = \frac{9}{4}$$

$$h = \frac{11}{2}$$

$$
\begin{aligned}
\text{Lat. } S &= 2\pi rh \\
&= \cancel{2} \times \frac{\overset{11}{\cancel{22}}}{7} \times \frac{9}{\underset{2}{\cancel{4}}} \times \frac{11}{2} \\
&= \frac{1089}{14} \\
&= 77.8 \text{ sq. in.} \\
\text{Lat. } S &= 77.8 \\
2 \text{ bases} &= 31.8 \\
\hline
\text{Total } S &= 109.6 \text{ sq. in.}
\end{aligned}
$$

NOTE: One must exercise one's judgment as to the advisability of using the $3\frac{1}{7}$ or 3.14 or 3.1416 as the value of π. It would be absurd to calculate the area of a tin can to four decimal places. Even finding it correct to one tenth of a square inch is impractical, for there must be some allowance for waste and seams.

D. HISTORY OF PI

It was not until the middle of the eighteenth century that the Greek letter π came into use as a symbol for the ratio of the circumference to the diameter of a circle. But from earliest times mathematicians knew that there was such a constant ratio. Different values were given it in different periods of history.

1. One of the earliest books we have is the Ahmes (Ah′mēz) Papyrus, written about 1700 B.C. In this manuscript a value is given to π equal to $\frac{256}{81}$ or 3.1604.

2. The Jews and Babylonians considered π equal to 3. This fact is shown in the measures given for sacred vessels in I Kings vii, 23 and II Chronicles iv, 2.

3. Archimedes of Syracuse, who lived between 287 and 212 B.C., was a great mechanical genius as well as mathematician. You will find it interesting to read the stories of his detection of the fraudulent goldsmith; his use of burning

glasses to destroy the Roman ships; his apparatus for launching ships; and the Archimedean screw used to drain the flooded fields of Egypt.

Archimedes proved that the value of π is between $3\frac{1}{7}$ and $3\frac{10}{71}$.

We can understand these values better by putting them in decimal form, but Archimedes did not have this advantage, because no one knew anything about decimal fractions until nearly 1600 A.D.

$$\left.\begin{array}{l} 3\frac{1}{7} = 3.1428 \\ 3\frac{10}{71} = 3.1408 \end{array}\right\rangle \pi = 3.14159+$$

4. Ptolemy, a great astronomer of Alexandria about 150 A.D., used $3\frac{17}{120}$ as the value of π. As a decimal $3\frac{17}{120} = 3.14166$.

5. Between 400 and 600 A.D. the Hindus used $\pi = 3$ or $3\frac{1}{8}$ and $\pi = \sqrt{10}$ which is 3.1622.

The Chinese had used $\pi = \sqrt{10}$ about 200 A.D.

6. The exact value of π cannot be expressed in ordinary figures, although many persons have contended long and earnestly that it could be done. If this were possible, a square could be constructed exactly equal to a circle. These people are known as "circle-squarers."

About 1600 the value of π was calculated to 35 decimal places. Since then it has been calculated to 707 decimal places, but it will never come out "even." In other words no square can be constructed that is *exactly* equal to a circle.

The value correct to the first 35 places is as follows:

$$\pi = 3.14159265358979323846246338327905288$$

When very exact measures are needed, we use $\pi = 3.14159$ or 3.1416.

For less exact measures, we use $\pi = 3.14$ or $\pi = 3\frac{1}{7}$.

E. PROBLEMS — CYLINDERS AND CIRCLES

1. Find the areas of the following circles:

(a) $r = 3\frac{1}{2}''$ (d) $r = 1$ ft. 8 in.

(b) $d = 5$ ft. (e) $d = 22''$

(c) $r = 6\frac{1}{2}$ cm. (f) $r = 150$ ft.

2. Find the formula for the area of a circle in terms of the diameter instead of the radius.

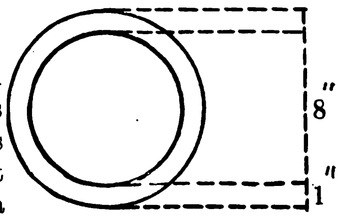
3. (a) Most modern houses and buildings are heated by passing steam or hot water through a set of cylindrical pipes called a radiator. The surface of these pipes is called the radiating surface, and its size depends upon the size of the room. It is measured in square feet.

(b) Find the amount of radiating surface from 12 pipes, $2''$ in diameter and $35''$ high. (Disregard the bases.)

(c) A radiator has two rows of ten $2''$ pipes $32''$ high. How many square feet of radiating surface has it?

(d) How much larger room will the second radiator heat than the first?

4. (a) The amount of water that flows through a pipe depends upon the area of its cross section, that is, upon the area of the circle inside the pipe.



(b) Find the area of a cross section of a pipe whose inner diameter is $8''$.

(c) The thickness of the iron of this pipe is $1''$. What is the diameter of the outer cross section? Find its area.

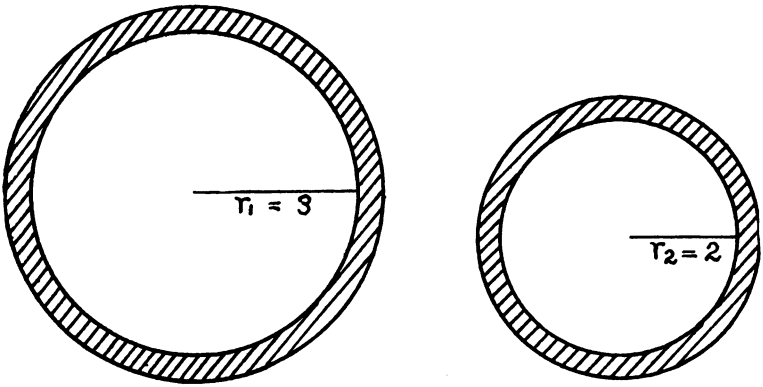
(d) From these two areas how may the area of the ring be found?

5. There is an easier way to find the area of a ring.

Let the outer circle be $\odot$ 1 with a radius of r_1 or 5, and the inner circle be $\odot$ 2 with a radius of r_2 or 4.

$$
\begin{aligned}
\odot 1 &= \pi r_1^2 \\
\odot 2 &= \pi r_2^2 \\
\hline
\text{Ring} = \odot 1 - \odot 2 &= \pi r_1^2 - \pi r_2^2 \\
&= \pi (r_1^2 - r_2^2) && \text{By factoring} \\
&= \pi (5^2 - 4^2) && \text{By substitution} \\
&= \pi (25 - 16) \\
&= \pi 9 \\
&= 3.1416 \times 9 \\
&= 28.2744 \text{ sq. in.}
\end{aligned}
$$

6. (a) The radius of the inner cross section of the first pipe is 3 in. and of the second 2 in. How



many times larger is the first than the second?

(b) *Solution.*

$$\text{In } \odot\ 1,\ r_1 = 3$$
$$\text{In } \odot\ 2,\ r_2 = 2$$
$$\text{Area of } \odot\ 1 = \pi r_1^2$$
$$\text{Area of } \odot\ 2 = \pi r_2^2$$
$$\text{The ratio of these areas } = \frac{\pi r_1^2}{\pi r_2^2}, \text{ or}$$
$$\text{by reducing to lowest terms } = \frac{r_1^2}{r_2^2}$$
$$= \frac{3^2}{2^2}$$
$$= \frac{9}{4} \text{ or } 2\tfrac{1}{4}$$

$\therefore \odot$ 1 is $2\frac{1}{4}$ times as large as $\odot$ 2.

(c) Do you have to compute the exact areas of two circles to find the ratio of their areas?

(d) The areas of two circles have the same ratio as the squares of their radii or as the squares of their diameters.

Show why radii or diameters may be used in these ratios.

7. (a) The rate of the flow of water through a cylindrical pipe is proportional to the area of its cross section.

(b) Two pipes have 1″ and 2″ inside measurements, respectively.

$$\text{Then } \frac{\odot\ 1}{\odot\ 2} = \frac{1^2}{2^2}$$
$$= \frac{1}{4}$$

Therefore 4 times as much water will flow per minute through the second pipe as through the first.

8. How much more water will flow per minute through a 3″ pipe than through a 1″ pipe?

9. How much faster will water flow through a 4″ pipe than through a $1\frac{1}{2}$″ pipe?

10. In a park is a fountain in the center of a circular grass plot 300 ft. in diameter. A 10-ft. cement walk surrounds the plot. What is the area of the walk?

11. (a) If the equatorial diameter of the earth is 7924 miles, how big is the equator?

(b) How many miles long is 1° at the equator?

12. (a) A company advertises for bids for painting 9-ft. bands around telephone poles whose average diameter is 14″.

(b) Mr. A figures paint at $3.00 per gallon and allows 1 gallon to 275 sq. ft. He figures 1 hour's time for painting a square and 7 hours' extra time per *C* poles for moving material from one to another. The labor costs $.70 per hour.

(c) Mr. B makes a bid of $65 per *C* poles, with all materials furnished.

(d) To which man should the company give the contract? How much is saved thereby?

13. (a) An oatmeal box is 7″ high and has a diameter of $4\frac{1}{4}$″. How large must be the paper used for the label around it?

(b) How much cardboard is needed for the box?

14. (a) A box of Dutch Cleanser is $4\frac{3}{4}$″ high. Its diameter is 3″. How large is the label covering the side?

(b) How large is the tin in each end?

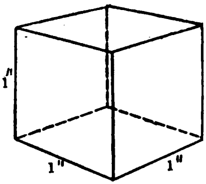
CHAPTER THIRTEEN

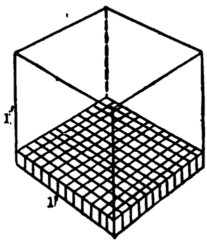
VOLUME

A. VOLUME OF A PRISM

1. A cube 1 cm. long is a cubic centimeter.

2. A cube 1″ long is a cubic inch.
A cube 1′ long is a cubic foot.





3. (a) How many 1″ cubes can be laid on the bottom of a cubical box 1 ft. long?

(b) It is evident that there can be one cube for each square inch of surface of the bottom.

(c) Since 1′ = 12″, there can be 12 × 12 or 144 cubes in one layer.

(d) Since the cube is 12″ high, how many layers of cubes can be put in?

(e) Evidently the total number of small cubes in the box is 12 times 144 or 1728.

Therefore 1 cu. ft. = 12^3 = 1728 cu. in.

4. Suppose the box were only 8″ high. The number of cubes would be 8 × 12 × 12.

5. Suppose the box were 9″ wide. Then each layer would have 12 × 9 cubes.

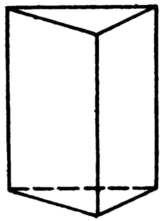
6. If the box were 12″ long, 9″ wide, and 8″ high, the number of cubes would be 12 × 9 × 8. Such a box is a *rectangular prism.*

7. The number of cubic inches the box can contain is called its *volume.*

Volume of other boxes may be measured in cubic centimeters, cubic feet, or cubic yards.

8. If l = length
w = width
and h = height of a rectangular solid, and V = volume,

then $V = lwh$.

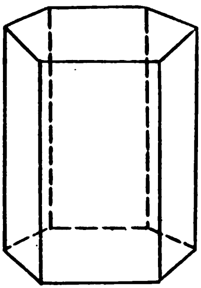


TRIANGULAR PRISM

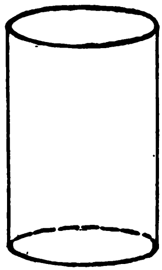
9. If the base of this box were a triangle, one cube could be placed on each square inch of the base, and the volume would be the area of the base $\times$ height.

$$V = Bh$$

10. The same statement will hold true if the base is a hexagon, other polygon, or a circle.



HEXAGONAL PRISM



CYLINDER

B. VOLUME OF A CYLINDER

1. (a) In each of these solids the area of the base, B, will have to be computed separately according to its shape.

(b) For a cylinder, we know the area of the base is πr^2. Therefore πr^2 may be substituted for B, giving

$$V\text{ cyl.} = \pi r^2 h.$$

2. (a) To find the area of any surface, both dimensions must be measured in the same unit of length. To find the volume, the three dimensions must be in the same unit of length.
   (b) How many dimensions must be known to find the volume of a cylinder? What are they?

C. PROBLEMS

1. (a) How many cu. ft. of water will a tank hold that is $8' \times 6' \times 5'$ inside measurement?
   When dimensions are given as above, it is understood that they are given in the order of l, w, and h.
   (b) How many gallons of water are in the tank when full? Allow $7\frac{1}{2}$ gallons to the cubic foot.
   (c) How many gallons are in the tank when the water is 2 feet deep?
   (Be sure you use the shortest method for your computation. Compare with 1 (a)).
   (d) What is the weight of the water in the tank when full? One cubic foot of water weighs 62.5 lb., or in other words, the density of water is 62.5 lb. per cu. ft.

NOTE: We have seen that it often pays not to perform a multiplication or division until it is absolutely necessary. Take such a problem as to find the number of gallons in a container $16'' \times 12'' \times 15''$. This problem may be solved

(1) By finding the number of cu. in., dividing by 1728, and multiplying by $7\frac{1}{2}$.

(2) By reducing each dimension in inches to feet, multiplying together and by $7\frac{1}{2}$.

(3) By finding the volume in cu. in. and dividing by 231, the exact number of cu. in. in one gallon.

Allowing $7\frac{1}{2}$ gallons to the cu. ft. is the same as using 230 cu. in. to the gallon.

(4) Or, the plan of No. (1) may be used, but each process indicated and then all performed at one time.

For comparison the two plans are given below.

Plan No. (1)

$$V = 16 \times 12 \times 15$$

$$= 2880 \text{ cu. in.}$$

$$2880 \text{ cu. in.} = \frac{2880}{1728} \text{ cu. ft.}$$

$$= 1\tfrac{2}{3} \text{ cu. ft.}$$

$$\begin{array}{r} 1\tfrac{2}{3} \\ 1728\overline{)2880} \\ \underline{1728} \\ 1152 \end{array}$$

$$\frac{1152}{1728} = \frac{288}{432} = \frac{36}{54} = \frac{2}{3}$$

$$\text{No. of gal.} = 7\tfrac{1}{2} \times 1\tfrac{2}{3}$$

$$= \frac{\cancelto{5}{15}}{2} \times \frac{5}{\cancel{3}}$$

$$= \frac{25}{2}$$

$$= 12\tfrac{1}{2} \text{ gal.}$$

Plan No. (4)

$$V = 16 \times 12 \times 15 \text{ cu. in.}$$

$$= \frac{16 \times 12 \times 15}{12 \times 12 \times 12} \text{ cu. ft.}$$

$$\text{No. of gal.} = \frac{\cancelto{\cancelto{2}{8}}{16} \times \cancel{12} \times \cancelto{5}{15}}{\cancelto{\cancelto{2}{4}}{12} \times \cancel{12} \times \cancelto{\cancel{4}}{12}} \times \frac{\cancelto{5}{15}}{\cancel{2}}$$

$$= \frac{25}{2}$$

$$= 12\tfrac{1}{2} \text{ gal.}$$

2. A mason jar has a diameter of $3\frac{1}{2}''$. It is $6''$ high. Show that it is exactly a quart measure.

3. A half-pint measuring cup is $2\frac{3}{4}''$ in diameter and $2\frac{1}{2}''$ high. Is it exactly a half-pint measure?

4. (a) For proper ventilation the law in most states requires at least 200 cu. ft. of air per pupil in a schoolroom.

(b) In a schoolroom $36' \times 24' \times 12'$, what is the largest number of pupils that should be enrolled?

(c) If the ceiling were two feet lower, what difference should be made in the enrollment of the room?

5. Measure your schoolroom to see how many pupils it can safely accommodate.

CHAPTER FOURTEEN

REVIEW OF FORMULAS

A. TRANSLATION OF FORMULAS

1. $P_{\square} = 4\,e$
2. $P_{\square} = 2\,(l + w)$
3. $S_{\square} = e^2$
4. $S_{\square} = l \cdot w = bh$
5. Lat. $S_{\text{cu.}} = 4\,e^2$
6. Tot. $S_{\text{cu.}} = 6\,e^2$
7. Lat. $S_{\text{ob.}} = 2\,h\,(l + w)$
8. Tot. $S_{\text{ob.}} = 2\,lw + 2\,lh + 2\,wh$
9. $S_{\triangle} = \frac{1}{2}\,bh$
10. Rt. $\triangle$: $c = \sqrt{a^2 + b^2}$
11. $S_{\text{trap.}} = \frac{1}{2}\,h\,(b_1 + b_2)$
12. $C_{\odot} = 2\pi r$
13. $S_{\odot} = \pi r^2$
14. Lat. $S_{\text{cyl.}} = 2\pi rh$
15. $V_{\text{cu.}} = e^3$
16. $V_{\square\,\text{pr.}} = lwh$
17. $V_{\text{pr.}} = Bh$
18. $V_{\text{cyl.}} = \pi r^2 h$

B. FORMULAS GIVEN IN THE SUPPLEMENT

1. Hero's formula for the area of a triangle.

$$S = \sqrt{s\,(s-a)(s-b)(s-c)}$$

2. Lat. $S_{\text{pyr.}} = \frac{1}{2}\,Pl$
3. $V_{\text{pyr.}} = \frac{1}{3}\,Bh$
4. Lat. $S_{\text{co.}} = \pi rl$
5. $V_{\text{co.}} = \frac{1}{3}\pi r^2 h$
6. $S_{\text{sph.}} = 4\pi r^2$
7. $V_{\text{sph.}} = \frac{4}{3}\pi r^3$

SUPPLEMENTARY TOPICS

A. INSPECTION METHOD OF FINDING SQUARE ROOT

The third method of finding square root is by inspection. By knowing the squares of numbers to 30, one can find by inspection the square root of certain numbers correct to one or two decimal places.

1. *Example:* Find the square root of 5.

$$
\begin{array}{llll}
 & 529 = 23^2 & \text{Diff. } 29 & \\
\sqrt{5} = \sqrt{5.00} = \ldots\ldots\ldots\ldots & & 2.2+ & = 2.23+ \\
 & 484 = 22^2 & \text{Diff. } 16 &
\end{array}
$$

Explanation: Annex two decimal ciphers to 5. Disregard the decimal point for the moment and consider the number 500. The square next higher is 529 or 23^2. The one next lower is 484 or 22^2. Then the $\sqrt{500}$ must be between 22 and 23, i.e. 22 +. By putting the decimal point in again, we get $\sqrt{5.00} = 2.2+$.

A little practice in estimation will give the next digit. The difference between 484 and 500 is 16; between 500 and 529 is 29. These show that 500 is less than half way between the two known squares. Therefore, the next digit in the root is less than 5. By comparing the difference, one can estimate the root to be 2.23 +.

Extract the root and compare results.

2. Another example. $\sqrt{8} = ?$

$$
\begin{array}{llll}
 & 841 = 29^2 & \text{Diff. } 41 & \\
\sqrt{8} = \sqrt{8.00} = \ldots\ldots\ldots\ldots & & 2.8+ & = 2.82+ \\
 & 784 = 28^2 & \text{Diff. } 16 &
\end{array}
$$

Extract the root and compare results.

3. By inspection, find the square roots correct to two decimal places of 2, 3, 6, and 7.

4. Find correct to one decimal place the square roots of 175, 150, 205.

5. A square field contains 10 acres. What is the length of each side? What would wire fencing for it cost at 90¢ per rod?

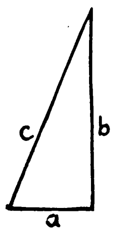
6. By inspection, find correct to one decimal place the square roots of:

(a) 80	(e) 56	(i) 135
(b) 180	(f) 45	(j) 700
(c) 32	(g) 108	(k) 535
(d) 105	(h) 600	(l) 1000

B. HERO'S FORMULA FOR FINDING THE AREA OF A TRIANGLE

1. Sometimes it is easier to measure the three sides of a triangle than to measure the base and altitude. This is true of any plot of ground that has some obstruction in the center, as a house or pond or swamp.

About 100 B.C. a Greek surveyor at Alexandria, Egypt, found a way to measure a triangular field from the length of the three sides. The surveyor's name was Hero and the formula is known as Hero's formula.



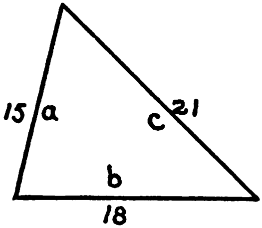
2. If the sides of a triangle are 5″, 12″, and 13″, he takes half the sum and calls it s, which is 15. From this half sum he subtracts each side in succession, getting the remainders 10, 3, and 2. He multiplies these remainders by the half sum and extracts the square root of the product.

$$\begin{aligned} s &= \tfrac{1}{2}(a+b+c) \\ &= \tfrac{1}{2}(5+12+13) \\ &= \tfrac{1}{2} \times 30 \\ &= 15 \end{aligned} \qquad \begin{aligned} s-a &= 15-5=10 \\ s-b &= 15-12=3 \\ s-c &= 15-13=2 \end{aligned}$$

$$s\,(s-a)\,(s-b)(s-c) = 15 \cdot 10 \cdot 3 \cdot 2$$

$$\begin{aligned} S_{\triangle} = \sqrt{s\,(s-a)\,(s-b)\,(s-c)} &= \sqrt{15 \cdot 10 \cdot 3 \cdot 2} \\ &= \sqrt{900} \\ &= 30 \text{ sq. in.} \end{aligned}$$

3. Instead of multiplying all these numbers together, it is easier to factor them, thus:



$$\begin{aligned} \sqrt{15 \cdot 10 \cdot 3 \cdot 2} &= \sqrt{3 \cdot 5 \cdot 2 \cdot 5 \cdot 3 \cdot 2} \\ &= \sqrt{3^2 \cdot 5^2 \cdot 2^2} \\ &= 3 \cdot 5 \cdot 2 \\ &= 30 \end{aligned}$$

4. Another illustration.

Given $a = 15''$
$b = 18''$
$c = 21''$

Find S.

$$S = \sqrt{s\,(s-a)(s-b)(s-c)}$$

$$\begin{aligned} s &= \tfrac{1}{2}\,(a+b+c) \\ &= \tfrac{1}{2}\,(15+18+21) \\ &= \tfrac{1}{2}\,(54) \\ &= 27 \end{aligned}
\qquad
\begin{aligned} s-a &= 27-15 = 12 \\ s-b &= 27-18 = 9 \\ s-c &= 27-21 = 6 \end{aligned}$$

$$S\,(s-a)(s-b)(s-c) = 27 \cdot 12 \cdot 9 \cdot 6$$

$$\begin{aligned} S_{\triangle} = \sqrt{s(s-a)(s-b)(s-c)} &= \sqrt{27 \cdot 12 \cdot 9 \cdot 6} \\ &= \sqrt{3 \cdot 3 \cdot 3 \times 3 \cdot 2 \cdot 2 \times 3 \cdot 3 \times 2 \cdot 3} \\ &= \sqrt{3^2 \cdot 3^2 \cdot 2^2 \cdot 3^2 \cdot 2 \cdot 3} \\ &= 3 \cdot 3 \cdot 2 \cdot 3\sqrt{2 \cdot 3} \\ &= 54\sqrt{6} \\ &= 54 \times 2.449 + \\ &= 132.24 + \text{ sq. in.} \end{aligned}$$

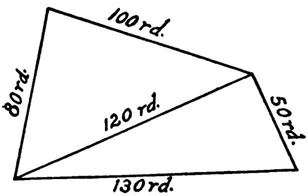
or

$$\sqrt{27 \times 12 \times 9 \times 6} = \sqrt{17496}$$

$$S_{\triangle} = 132.2 +$$

$$\begin{array}{rl}
 & \overline{1}\,\overline{74}\,\overline{96}.\overline{00}\ \underline{|\,132.2 +} \\
 & 1 \\
23 & |\,\overline{74} \\
 & \ \ 69 \\
262 & |\,\overline{596} \\
 & \ \ 524 \\
2642 & |\,\overline{7200} \\
 & \ \ 5284 \\
 & |\,1916
\end{array}$$

5. Find the area of a triangle whose sides are 5, 6, and 7 inches respectively.



6. Find $S_{\triangle}$ if $a = 9$, $b = 10$, and $c = 11$ ft.

7. An irregular field has its successive sides 130, 50, 100, and 80 rods. Its shorter diagonal is 120 rods. What is the area of the field?

C. LESS COMMON SOLIDS

There are three other solids which we shall examine briefly because, although important, they are less common than others. They are the pyramid, cone, and sphere. Only the right pyramid and right circular cone will be considered.

I. The Pyramid

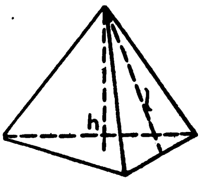
1. How many bases has a pyramid?

2. What shapes may the base have?

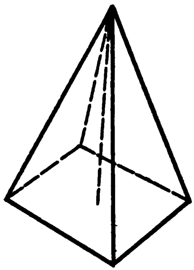
3. What is the shape of the lateral faces?

4. The point at which the lateral faces meet is the *vertex* of the pyramid.

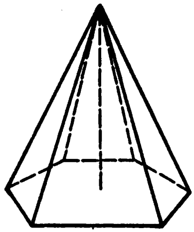
5. The altitude of one of the triangular faces is called the *slant height* of the pyramid (l).



TRIANGULAR PYRAMID



RECTANGULAR PYRAMID



HEXAGONAL PYRAMID

6. A perpendicular line from the vertex to the center of the base is the *height* of the pyramid or its *altitude.*

7. If possible find a hollow prism and a hollow pyramid with equal bases and equal height. Use the pyramid as a measuring cup to fill the prism with water.

You will find that the pyramid must be filled three times.

$$\text{For the prism, } V_{\text{pr}} = Bh$$
$$\text{for the pyramid, } V_{\text{pyr}} = \tfrac{1}{3}\, Bh$$

To show the difference between V for prism and V for pyramid they may be written V_{pr} and V_{pyr} respectively.

8. A rectangular pyramid has a base 8″ square and a height of 12″. Find its volume.

9. The base of a pyramid is an equilateral triangle whose edge is 4″. The slant height is 6″. Find the lateral area.

10. The area of the base of a pyramid is 20 sq. in.; the altitude is 15 in. Find its volume.

11. The base of a pyramid is a regular hexagon 8 ft. on a side; the height is 12 ft. Find its lateral area.

12. What is the shape of each face of a pyramid? How can you find the area of one face from the slant height and the edge of the base?

13. The base of a regular hexagonal pyramid is 6″ on a side. The slant height is 10″. Find the lateral area.

14. The base of a pyramid contains 75 sq. in. Its height is 12 in. What is its volume?

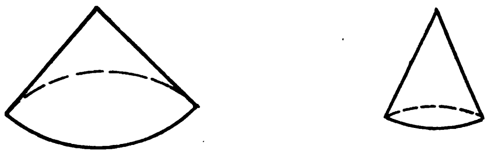
II. The Cone

1. Draw and cut out a circle.

2. Along one radius cut from the circumference to the center.

3. Lap over the two edges, at first just a little, then one-fourth to one-half and until the two edges meet.

4. The resulting figures are cones.



5. You can readily see that the lateral surface of a cone is a part of a circle.

If the radius of the base is r and the slant height of the cone is l, then the

$$\text{Lat. } S_{co} = \pi rl.$$

6. Just as the pyramid is $\frac{1}{3}$ of the prism, so the cone is $\frac{1}{3}$ of the cylinder having the same base and height.

$$\text{Since } V_{cyl} = \pi r^2h$$
$$V_{co} = \tfrac{1}{3}\pi r^2h.$$

7. A round tower 21 feet high is 10 feet in diameter and is capped by a cone 12 feet high.

(a) The slant height is the hypotenuse of a right △. Find its length.

(b) How many square feet of tin are required to cover the roof?

(c) What will it cost to paint the tower at $2.25 per square, and to paint the roof at $2.00 per square?

8. On a barn floor is a pile of wheat. Naturally its shape is a low cone. It is 8′ in diameter and $2\frac{1}{4}'$ high. How many bushels of wheat are in the pile if $1\frac{1}{4}$ cu. ft. make 1 bushel?

9. The slant height of a cone is $4\frac{1}{2}$ ft. The radius of the base is 2 ft. Find the lateral area.

10. Find the total area of a cone whose slant height is 3″ and whose base has a radius of 2″.

11. A cone 6 in. high has a base with a 4-in. radius. What is its volume?

III. The Sphere

Sphere is the mathematical name of the toy of your earliest childhood. If a sphere is cut in two equal parts, two hemispheres (half-spheres) are formed.

1. The surface of a sphere is four times as large as a circle with the same diameter.

$$S_{\text{sph}} = 4\pi r^2$$

2. The volume of a sphere is the cube of the radius multiplied by $\frac{4}{3}\pi$.

$$V_{\text{sph}} = \frac{4}{3}\pi r^3$$

3. A steel ball is 10″ in diameter.
   (a) How many cu. in. are in its volume?
   (b) Find its weight if the density of steel is 28 lb. per cu. in.

4. In round numbers the diameter of the earth is 8000 miles.
   (a) In round numbers find the total surface of the earth.
   (b) What is the ratio of the land area to the water area?
   (c) How many square miles of land are there?
   (d) In round numbers find how many cubic miles the earth contains.

5. Find the surfaces of the following spheres:

   (a) $r = 3\frac{1}{2}''$
   (b) $r = 4\frac{1}{5}''$
   (c) $r = 5$ ft. 3 in.
   (d) $r = 3\frac{3}{8}''$

6. Find the volume of each sphere given in example 5.

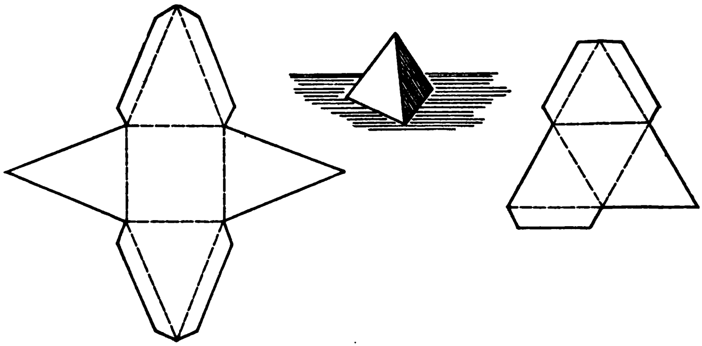
IV. Patterns for Solids

1. What is a pyramid?

How does a rectangular pyramid differ from a triangular pyramid?

2. Polyhedron is a name given to any solid bounded by

plane faces. This name comes from two Greek words, *poly*, which means *many*, and *hedron*, which means *base* or *face*.



TETRAHEDRON

Name all the polyhedrons that you know.

The faces of a regular polyhedron are regular polygons.

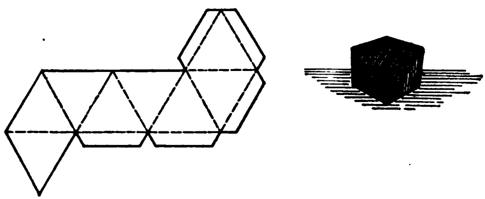
3. The regular polyhedron having four equilateral triangles as faces is sometimes called a *tetrahedron*.

Since *tetra* is the Greek word which means *four*, why is tetrahedron a good name for a triangular pyramid?

4. To construct these figures, draw the patterns on stiff paper, cut out, and fold on the dotted lines. Paste together with the flaps.

5. What kind of solid has square faces?

Why may a cube be named a *hexahedron?*



OCTAHEDRON

6. A solid having eight triangular faces is called an *octahedron*.

Which part of the name tells the number of faces?

Does this part mean the same number in octagon and octave?

7. There are only two other regular polyhedrons. One has twelve regular pentagons for faces; the other has twenty equilateral triangles.

MISCELLANEOUS PROBLEMS

1. A girl wishes to make a shirt waist box out of a canned goods box from a grocery. It measures $26'' \times 14'' \times 13''$. How much Japanese matting is needed to cover the outside of the box?

How much cretonne 27 in. wide is needed for lining the inside?

2. (a) How much paint is needed to paint four square columns 3 ft. wide and 15 ft. high? (See example 13, page 18.)

(b) The painter receives \$.65 per hour and takes 2 hours for a square. How much does he receive?

3. The walls and ceiling of a room are tinted. How much surface is covered if the room is $14' \times 14' \times 9'$? Allow for two doors $7' \times 3\frac{1}{3}'$ and one window $7' \times 4'$.

4. How much tin is needed for a 5-lb. candy box which is $10\frac{1}{2}'' \times 6\frac{3}{4}'' \times 3\frac{1}{2}''$? Allow $\frac{1}{2}$ inch for the overlapping of the lid.

5. A boy has made a tool chest $28'' \times 15'' \times 12''$. Another boy agrees to stain the sides and both sides of the lid for him at 2¢ per square foot. How much does the second boy earn?

6. How many square feet of surface are in the walls and ceiling of your living room at home? In the floor?

7. A Gold Dust box is 4″ long, $1\frac{1}{8}''$ wide, and $6\frac{1}{4}''$ high. How many square inches are in the paper label pasted all over it?

8. A box of pepper is $3\frac{7}{8}''$ high. The bottom is $1\frac{1}{2}'' \times 1''$.

(a) How much paper is used for the label pasted all around the sides?

(b) How much tin is used in the box?

9. A water tank is $3\frac{1}{2}$ ft. long, $2\frac{3}{4}$ ft. wide, and 2 ft. deep. How many square feet of zinc are required to line

the four sides and base, allowing 2 sq. ft. for overlapping and for turning the top edge?

10. (a) Just inside a rectangular garden $40' \times 32'$ is a walk $4'$ wide. How many square yards of cement pavement are in the walk?

(b) How many square feet of the garden are left for flowers?

(c) What part of the entire garden is used for flowers?

11. A pasture 120 ft. long and 100 ft. wide has an 8-ft. gate in one end. How much wire will be needed to fence it?

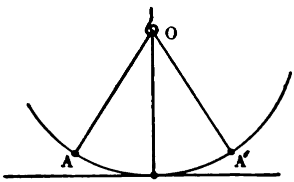
12. A picture $15''$ by $12''$ is to be framed with $1''$ molding. How many feet of molding will it take?

13. Material 50 inches wide was bought for draperies. It is discovered after the curtains are made that a piece 18 inches long is left, enough to make half of a valence $18'' \times 100''$. All the 50-inch material had been sold, but the same pattern in 30-inch material can be bought. How much of the 30-inch drapery is needed to finish one valence and to make another?

14. Find the square roots of the following:

(a)	5476	(b)	233289	(c)	5776
(d)	1849	(e)	532900	(f)	5041
(g)	374544	(h)	7225	(i)	9025
(j)	5625	(k)	7569	(l)	331776
(m)	5329	(n)	6084	(o)	356409
(p)	8649	(q)	736164	(r)	793881
(s)	4562496	(t)	18190225	(u)	39204

15. (a) What is a pendulum? How is it used?



(b) A pendulum swings from some fixed point of support as O, through the arc of a circle as AA'. The time it takes for the pendulum to swing

through this arc depends upon the length of the pendulum or OA.

(c) A formula has been found by which one can find the time it takes any pendulum to swing through its arc.

The formula is, $t = \pi \sqrt{\frac{l}{g}}$.

t is the time or number of seconds,
π is 3.1416 as used in a circle,
l is the number of feet in the length of the pendulum,
g is the force of the attraction of the earth or gravity. g has an approximate value of 32.

(d) Read this formula in English.

(e) How many seconds does it take a pendulum 4 ft. long to swing through its arc?

(f) Find the time for a 3-ft. pendulum; a 2-ft. pendulum; a 1-ft. pendulum.

16. (a) If a stone is dropped from a second story window 16 ft. above the ground and another is dropped from a window four times as high, 64 feet above the ground, do you think it takes four times as long for the second stone to reach the ground? That is the natural conclusion, but it is not true.

(b) There is a formula by which the time it takes a body to fall may be found.

The formula is, $t = \sqrt{\frac{2\,s}{g}}$.

t is the time in seconds,
s is the number of feet in the distance through which the body must fall,
g is the force of gravity, which is 32.

(c) What does $2\,s$ mean?

(d) Read the formula in English.
(e) How many seconds does it take a stone to fall 16 feet?
(f) Find the time it takes for one to fall 64 feet.
(g) Compare the two times.
(h) How long does it take a body to fall 8 feet?
(i) In a storm a ball was loosened on the top of a church spire 320 feet high and fell to the ground. How many seconds was it in falling?

17. The vertex angle of an isosceles triangle is 36°, what is the size of each base angle?

18. In a triangle ABC, angle B is twice angle A, and angle C is three times angle A. How many degrees are in each angle?

19. One acute angle of a right triangle is 28° 40′. What is the size of the other acute angle?

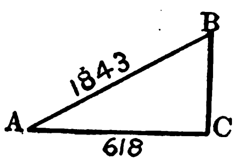
20. One angle of a triangle contains 50° 30′; another contains 88° 40′. What is the size of the third angle?

21. In a triangle ABC, $\angle A = 48° 50'$; $\angle B = 65° 30'$. What is the size of the exterior angle at vertex C?

22. A wire is fastened at one end to a telegraph pole, 18 ft. from the ground, and at the other to a stake at the level of the ground, 14 ft. 6 in. from the foot of the pole. How long is the wire?

23. A May pole, 10 ft. high, is set in a circle whose radius is 8 ft. How long must be the streamers fastened at the top of the pole in order that they may reach the edge of the circle?

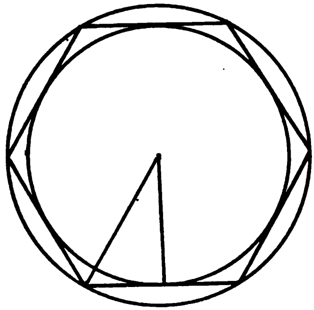
24. If the foot of a ladder 24 ft. long is 12 ft. from a house, how far up the side of the house does the ladder reach?



25. In a right triangle, $a = 67.2'$ and $c = 110'$. Find b.

26. From the data given in the figure, find a.

27. The sides of a triangle are 37.5 ft., 90 ft., and 97.5 ft. Classify the triangle as to its sides and angles.



28. (a) A regular hexagon is inscribed in a circle whose radius is 12 ft. Find the radius of the circle inscribed in the hexagon.

(b) What is the perimeter of the hexagon?

29. (a) A girl crochets a medallion in the shape of an isosceles triangle. It is to be 8 inches long at the base and 3 inches high. How long must each edge be?

(b) She wants two others, each 5 inches at the base and $1\frac{1}{2}$ inches high. How long is each side of the smaller ones?

30. A room is $18' \times 15'$. What is the shortest distance from one corner to the diagonally opposite one?

31. What is the largest square rod that can be cut out of $2\frac{1}{2}''$ round stock?

32. (a) Velvet, one yard wide, is cut on the bias. It is sold by the measure on the straight edge.

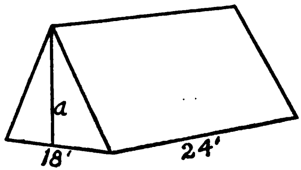
(b) How many inches on the straight must be purchased in order to have a bias band 8 inches wide?

(c) How long will the band be?

33. A balloon is 1500 ft. in the air. If a stone is dropped from it, how long will it take to reach the earth? Use formula

$$t = \sqrt{\frac{2\,s}{g}}\ .$$

34. (a) From the data given in the figure find the height of the tent pole, *a*.



(b) How many square yards of canvas are needed to make the tent?

35. (a) A girl has a flower bed in the shape of an equilateral triangle 8 ft. on a side. How many tulip bulbs must she buy if she allows 36 square inches for each tulip?

(b) What will they cost at 5 cents each?

36. City streets intersect in such a way that there is a triangular park formed 90 ft. on a side. How many square feet of sod are needed for it?

37. (a) In these circles hexagons and triangles are inscribed and triangles are circumscribed about them in two ways. Draw the figures and letter all the points of intersection.

(b) Find all the equal lines.

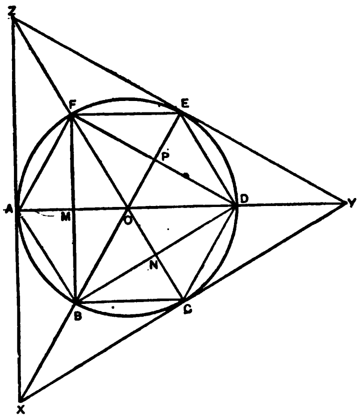
(c) Find all the equal triangles.

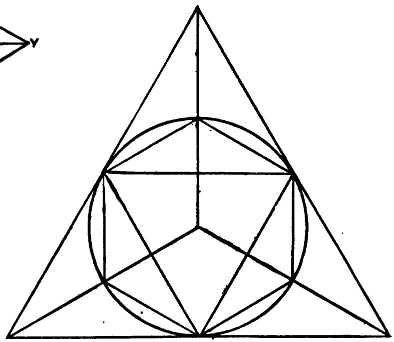
(d) Can you find any equal rhombi?

(e) How many degrees are in $\angle$ *ABC?* $\angle$ *FBD?* $\angle$ *ABF?* $\angle$ *CBD?*

(f) Without using the protractor find the number of degrees in each angle.

(g) Find all the equal angles.





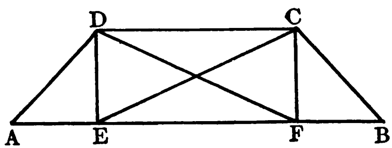
38. A ladder 38 ft. long is resting against a wall. If the foot of the ladder is 7 ft. 3 in. from the wall, how far above ground does the ladder reach?

39. A steel bridge has a truss *ABCD*.

$AB = 20$ ft. $ED = 6$ ft. 4 in.

$DC = 12$ ft.

Find the length of *AD* and *DF*.



40. (a) The following are dimensions and areas of rectangles. Find the ones missing and the perimeters.

	Area	Length	Width	Perimeter
(1)		$2\,ab + 5$	$ab + 3$	
(2)		$7\,a + 2\,b$	$3\,a + 4\,b$	
(3)	$6\,x^2y^2 + 14\,xy + 8$	$3\,xy + 4$		
(4)	$32\,\pi^2 + 88\,\pi R + 20\,R^2$	$8\,\pi + 2\,R$		
(5)	$135\,x^2 + 51\,xy + 2\,y^2$	$45\,x + 2\,y$		
(6)		$2\,\pi R + 6$	$2\,\pi R + 2$	
(7)	$12\,a^2 + 46\,ab + 42\,b^2$		$3\,a + 7\,b$	
(8)		$8\,x + 9\,y$	$7\,x + 3\,y$	
(9)	$2\,a^2 + 12\,a + 18$	$2\,a + 6$		
(10)	$28\,t^2 + 80\,tu + 48\,u^2$		$4\,t + 8\,u$	
(11)	$35\,x^2 + 34\,xy + 8\,y^2$		$5\,x + 2\,y$	
(12)	$36\,a^2 + 69\,ab + 30\,b^2$		$4\,a + 5\,b$	
(13)		$3a + 5b + 6c$	$2\,a + 2\,b + c$	
(14)		$81 + 3\,k$	$61 + 2\,k$	
(15)		$2\,x + 3\,y + 1$	$x + 2\,y + 3$	
(16)		$3\,a + b + 5\,c$	$2\,a + b + 3\,c$	
(17)		$3\,a + 5\,x + 6$	$2\,a + x + 7$	
(18)	$a^2 + 4b^2 + 9c^2 + 4ab + 6ac + 12bc$	$a + 2\,b + 3\,c$		
(19)		$\frac{1}{2}\,a + \frac{1}{5}\,b$	$\frac{1}{2}\,a + \frac{1}{5}\,b$	
(20)	$\frac{1}{4}\,x^2 + \frac{1}{3}\,xy + \frac{1}{9}\,y^2$		$\frac{1}{2}\,x + \frac{1}{3}\,y$	

(b) Which of these rectangles are squares?

41. A post 7 ft. high casts a shadow 4 ft. long at the same time of day that a tree casts one 32 ft. long. Find the height of the tree.

42. A boy 5 ft. tall makes a shadow 10 ft. long. At the same moment the shadow of a building is 125 ft. How high is the building?

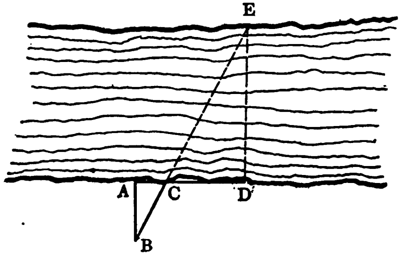
43. By the shadow method, measure the heights of trees or buildings in the vicinity of your school.

44. A and D are right angles.

If $AB = 33$ ft.

$AC = 15$ ft.

and $CD = 125$ ft.
what is DE, the width of the river?

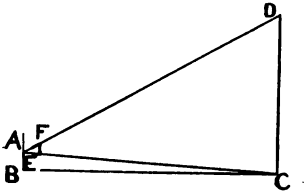


45. (a) Find the dimensions and perimeters of the following rectangles. Check results.

	Area	Length	Width	Perimeter
(1)	$\frac{1}{2}x^2 + \frac{1}{2}xy$	......	......	....
(2)	$\frac{1}{3}ax + \frac{1}{6}ay + \frac{1}{9}az$	......	......	....
(3)	$\frac{1}{2}a(m+n) + \frac{1}{4}b(m+n)$	......	......	....
(4)	$(x+1)(x+3) + (x+2)(x+3)$	......	......	....
(5)	$(a+b)(a+c) + (a+d)(a+c)$	......	......	....
(6)	$(a+2)^2 + 5(a+2)$	......	......	....
(7)	$\frac{1}{4}a^2 + \frac{1}{3}ab + \frac{1}{9}b^2$	......	......	....
(8)	$25x^2 + 65ax + 42a^2$	......	......	....
(9)	$10^2 + 6(10) + 9$	......	......	....
(10)	$196 + 28x + x^2$	......	......	....
(11)	$25a^2 + 60ay + 36y^2$	......	......	....
(12)	$63x^2 + 58xy + 7y^2$	......	......	....
(13)	$9^2 + 9(9) + 20$	......	......	....
(14)	$a^2 + \frac{2}{3}a + \frac{1}{9}$	......	......	....
(15)	$289a^2 + 136a + 15$	......	......	....
(16)	$6^2 + 12(6) + 35$	......	......	....
(17)	$x^2 + \frac{5}{6}x + \frac{1}{6}$	......	......	....
(18)	$(\frac{1}{3})^2 + 5(\frac{1}{3}) + 6$	......	......	....
(19)	$81x^2 + 72ax + 16a^2$	......	......	....
(20)	$4a^2 + 9b^2 + c^2 + 12ab + 4ac + 6bc$	......	......	....

(b) Which of these rectangles are squares?

46. (a) A boy wishes to measure the height of the school building. His only instruments are a tape line and a 15-in. ruler. He holds the ruler vertically in front of him, walks to such a position that he can just sight top and bottom of the building over the top and bottom of the ruler. A classmate measures for him his distance from the building and finds it is about 79.9 ft. The boy's eye is 5 ft. from the ground. The bottom of the ruler is 24 in. from his eye.



(b) Given $AB = 5$ ft.
$BC = 79.9$ ft.

To find AC to the nearest integer.

(c) What kind of triangles are AEF and ACD?

Given $AE = 24$ in.
$EF = 15$ in.

Find CD, the height of the building.

47. On a map Reims is on a line $7\frac{3}{4}''$ east of Paris and $2\frac{3}{4}''$ north of it. The map is drawn on a scale of $1''$ to 10 mi. What is the distance from Paris to Reims?

48. On a blue print drawn to a scale of $\frac{1}{8}''$ to a foot, what lengths are represented by the following lines?

(a) $1\frac{3}{4}''$	(d) $\frac{5}{16}''$	(g) $12\frac{3}{4}''$	(j) $3\frac{1}{8}''$
(b) $5\frac{5}{8}''$	(e) $7\frac{1}{4}''$	(h) $15\frac{3}{8}''$	(k) $1\frac{7}{16}''$
(c) $2\frac{3}{16}''$	(f) $10\frac{1}{2}''$	(i) $4\frac{7}{8}''$	(l) $8\frac{5}{16}''$

49. (a) If it takes 2 cu. yds. of gravel for a 30-ft. sidewalk, how much will it take for a 42-ft. walk?

(b) A sack of cement lays 1 ft. of sidewalk. If it is spread thin and 8 sacks are used for 9 ft. of walk, how much will it take for 18 ft.? For 20 ft.? For 30 ft.? For 42 ft.?

50. (a) A garage is 20′ by 18′ by 10′. How many squares are in its lateral area?

(b) The roof is in the shape of two triangles at the ends and two trapezoids at the sides of the garage. If it extends 2 ft. over the sides of

the building, what is the length of the base of the triangular parts of the roof? Of the trapezoidal parts?

(c) The sloping edge (hip rafter) is 16 ft. long. The ridge pole is 2 ft. long. What is the altitude of each triangular section? of each trapezoidal section?

(d) How many square feet of roofing are needed?

51. A baking powder box is 5″ high. Its diameter is 3″. What is the size of the label?

Allow $\frac{1}{2}$″ for overlapping of the lid and find how much tin the box takes.

52. A piece of stove pipe is worn out. It is 23″ long and $4\frac{1}{2}$″ in diameter. How much sheet iron is needed for a new piece?

53. A cylindrical salt box is $5\frac{5}{8}$″ high and $3\frac{3}{8}$″ in diameter. How large must the label be to surround the box? How much cardboard is needed for the whole box?

54. Measure some object near your school which is cylindrical in shape. Calculate the lateral and total areas.

55. Make the same measurements and calculation for some cylindrical object in your home or place where you work.

56. What is the area of a watch crystal if its diameter is $1\frac{3}{4}''$?

57. A round rattan work basket has a radius of $3\frac{1}{2}''$. What is the area of the top of the lid?

What is the total area of the lid, if it extends $1\frac{1}{2}''$ over the basket?

58. A preserving kettle is 13″ in diameter. How much tin is needed for a lid to cover it?

59. (a) A pint cup is $2\frac{1}{2}$ inches high and 11 inches around. How much tin is needed for it?

(b) Show that it holds a full pint.

60. A measuring cup is 6 cm. high and $22\frac{1}{4}$ cm. around. How much aluminum is needed for the cup?

REVIEW QUESTIONS

On Chapter One

1. Define the following terms: parallel, perimeter, formula, coefficient.

2. Describe a cube; an oblong block.

3. Give the formula for the perimeter of a square; of a rectangle.

4. Of what use is the process of substituting numerical values in a formula?

On Chapter Two

1. Define the following terms: angle, right angle, straight angle, perpendicular, acute angle, obtuse angle, adjacent sides, surface, area, lateral area, the square of a number, the square root of a number, factor, prime factor, exponent.

2. Inquire from a carpenter and draftsman the uses of a T square.

3. Describe the following: square, rectangle, parallelogram, rhombus, rhomboid, radical sign.

4. Give the formula for the area of a square; of a rectangle.

5. How does a decimal fraction differ from other fractions?

6. Is the square of a number greater or less than its square root?

On Chapter Three

1. Define the following terms: diagonal, vertex, complementary angles, interior angles of a triangle, exterior angles of a triangle, hypotenuse, leg of a right triangle, legs of an isosceles triangle.

2. What is the sum of the interior angles of a triangle?

3. Classify triangles according to the size of their angles. Describe each kind.

4. Classify triangles according to the relative lengths of the sides. Describe each kind.

5. Give the formula for the area of a triangle.

6. Two triangles have two sides of one equal respectively to two sides of the other. In the first triangle these sides form an angle of 90°, and in the second an angle of 60°. Which triangle is the larger? Why?

On Chapter Four

1. Define the following terms: circle, diameter, semicircle, circumference, arc, intersect, base angles and vertex angle of an isosceles triangle, bisect, bisector.

2. Describe an angle inscribed in a circle; one inscribed in a semicircle.

3. A square and an octagon are inscribed in a circle. Which has the longer perimeter? Why?

4. Why are the maps on ordinary railroad time tables distorted? Are the real railroads in as straight lines as the maps represent them?

On Chapter Five

1. Define the following term: circumscribed circle.

2. Describe all the special features of an isosceles triangle and its altitude.

3. Who was Pythagoras? What is the Pythagorean theorem?

On Chapter Six

1. Define the following terms: parallel lines, transversal, interior angles of parallel lines, exterior angles of parallel lines, alternate interior angles, corresponding angles, vertical angles, supplementary angles.

2. When two parallel lines are cut by a transversal, which angles are equal and which are supplementary?

3. If the vertex of a triangle is folded over to the foot of a perpendicular drawn from that vertex, what is the relation of the fold to the base of the triangle?

On Chapter Seven

1. Define the following terms: median, midjoin.

2. In a parallelogram which lines and which angles are equal?

3. Draw the two diagonals and answer the above question.

4. How do the answers to these questions change when the parallelogram is a rhombus? a rectangle? a square?

On Chapter Eight

1. Define the following terms: ratio, equation, graph, monomial, binomial, trinomial, polynomial.

2. Explain percentage in terms of ratio.

On Chapter Nine

1. How does the sum of the interior angles of a rhomboid compare with the sum of those of a rectangle?

2. What is the formula for the diagonal of a square in terms of its side?

3. Of what general formula is that for the diagonal a special case?

On Chapter Ten

1. Define the following terms: similar figures, proportion.

2. What are some of the practical uses of similar figures?

3. Under what conditions are two triangles similar?

4. Give all the interesting facts you can about Thales.

5. Which man do you think did the most for mathematics, Thales or Pythagoras? Why?

6. Describe the following instruments and their uses: transit, sextant, quadrant, baculus.

On Chapter Eleven

1. Describe the following figures: trapezoid, isosceles trapezoid.

2. Give the formula for the area of a trapezoid.

3. How is this formula related to that for the area of a triangle?

4. Examine as many geometrical designs as possible in linoleum, wall paper, etc., and see if certain polygons are used much more than others. List the polygons in order of the frequency of their use.

On Chapter Twelve

1. Define the following terms: concentric circles, cylinder, cylindrical surface, plane surface.

2. What is the test for a plane surface?

3. What is the ratio of the diameter to the circumference of a circle? Give a brief history of this ratio.

4. What is the formula for the circumference of a circle? For the area of a circle?

On Chapter Thirteen

1. Describe a prism.

2. Describe triangular, rectangular, and hexagonal prisms.

3. What is the difference between a cylinder and a prism?

SYMBOLS AND ABBREVIATIONS

$\parallel$ = parallel, is parallel to.
P = perimeter.
B = area of base.
b = base (base line).
Lat. = lateral.
e = edge.
l = length.
w = width.
h = height.
□, $\boxed{s}$ = square, squares.
▭, $\boxed{s}$ = rectangle, rectangles.
▱, ▱s = parallelogram, parallelograms.
▱, ▱s = rhombus, rhombi.
$\triangle$, $\triangle$s = triangle, triangles.
$\angle$, $\angle$s = angle, angles.
⏢ = trapezoid.
rt. = right.
S = surface or area of surface.
$\sqrt{\ }$ = square root of.
comp. = complementary.
sup. = supplementary.
hyp. = hypotenuse.
alt. = altitude.
$\odot$, Ⓢ = circle, circles.
isos. = isosceles.
⌒, ⌒s = arc, arcs.
$\therefore$ = therefore.
$\perp$, $\perp$s = perpendicular, or is perpendicular to, perpendiculars.
a, b, c = sides of scalene triangle.
a, b = legs of right triangle.

c = hypotenuse of right triangle.
( ) = parentheses.
$\sim$ = similar, is similar to.
π = pi, ratio of circumference to diameter of circle.
V = volume.
c = circumference.
d = diameter.
r = radius.
cu. = cube.
co. = cone.
cyl. = cylinder.
pr. = prism.
pyr. = pyramid.
sph. = sphere.

APPENDIX

A. MATHEMATICS CLUBS

Mathematics clubs among students in colleges and high schools have proved their worth. Nearly all of the progressive schools of these grades number such organizations among their student activities.

It was the author's privilege to organize and direct a mathematics club among the boys of grades ten to twelve in the high school in which she formerly taught mathematics. Three years ago, she suggested, as an experiment, the organization of a similar club in one of the junior high schools of Columbus. Under the direction of a mathematics teacher, the first club of ninth grade pupils has become the Alpha Chapter of the original Euclidean Club, for the Beta and Gamma Chapters have been organized in the eighth and seventh grades respectively.

The purpose of a mathematics club is to promote interest in the study of mathematics, to give the pupils glimpses of the future, which serve as incentives to continue the study, and to furnish an outlet for their social instincts.

Stories from the history of mathematics, magic squares and circles, mathematical fallacies, and other recreations furnish interesting material for club programs.

The topics may be the same for clubs in the several grades but the treatment will be different in each. To show this difference a program for each grade in the same topic is given:

Topic — Magic Squares.

I. Ninth Grade.

1. History of Magic Squares.

2. How to Make Magic Squares (with an odd number of sides).
3. How to Make Magic Circles.

II. Eighth Grade.

1. How to Make a Magic Square (with odd number of sides).
2. How to Make a Magic Circle.
3. Some Interesting Facts about Magic Squares.

III. Seventh Grade.

1. A Magic Square, 3 numbers on a side.
2. A Magic Square, 5 numbers on a side.
3. A Magic Square, 7 numbers on a side.
4. Board Work with Magic Squares.

These sample programs for the ninth grade may be suggestive:

I. 1. Euclid.
2. Some Interesting Things about a Billion. (White.)
3. How to Write 100 in Several Ways. (T. C. Record, November, 1912.)
4. Some Questions. (Original by pupil.)

II. 1. Familiar Trick with Dice. (White.)
2. Mathematical Advice to a Building Committee. (White.)
3. Puzzle of the Camels. (White.)
4. Pythagoras.

III. 1. Multiplication on Fingers.
2. Russian Multiplication (only table of 2's need be known). (School Science and Mathematics, April, 1919.)
3. Ship Carpenter's Puzzle. (White.— Presented one meeting. Solution given the next.)
4. Story of Flatland (told by a pupil).

Topics taken from the following subjects will be interesting for other programs:

1. Napier's Rods.
2. Descartes' Life.
3. A Fairy Tale. (School Science and Mathematics.)
4. A Number Trick. (White.)
5. A Riddle. (Jones.)
6. To Prove 1 = 2.
7. Poem: A Young Lady and Her Lover. (Jones.)
8. Some Interesting Questions. (Jones.)
9. Remarkable Numbers. (Teachers' College Record, November, 1912, or Jones.)
10. Trisecting an Angle.
11. Duplicating the Cube.
12. Squaring the Circle.
13. Fourth Dimension.
14. Mathematical Symbolism.
15. The Golden Section. (May, 1918, of American Mathematical Monthly.)
16. Proofs of Pythagorean Theorem. (Monograph — D. C. Heath & Co.)
17. Use of Mathematics in Science
18. History of Arithmetic.
19. History of Algebra.
20. What is a Straight Line?
21. Computing Machines.
22. History of Pi.
23. The Algebra of Al-Khowarizmi.
24. Hindu-Arabic Numerals.
25. Paper Folding.
26. Fallacies of Arithmetic.
27. Opportunities Open to Students of Mathematics.
28. Women Mathematicians. (March, 1918, of A. M. M.)
29. Game of "Nim." (March, 1918, of A. M. M.)

30. Chinese Rings. (March, 1918, of A. M. M.)
31. A, B, and C. The Human Element in Mathematics. (S. Leacock's "Literary Lapses.")
32. Logarithms.
33. The Oldest Mathematical Work.
34. Great Mathematicians, as:
    (a) Newton, Astronomer and Mathematician.
    (b) Archimedes, Inventor and Mathematician.

Pupils become so interested in the mathematics clubs that they bring in material, invent games and tricks, and even write and dramatize mathematical plays.

The following books and magazines are helpful:

1. Abbott. Flatland. Little, Brown, $0.60.
2. Andrews. Magic Squares and Cubes. Open Court Pub. Co., $1.50.
3. Ball. History of Mathematics. Macmillan, $3.25.
4. Ball. Primer of the History of Mathematics. Macmillan, $0.60.
5. Ball. Mathematical Recreations. Macmillan, $2.25.
6. Jones. Mathematical Wrinkles. S. P. Jones of Gunter, Texas, $1.65.
7. Smith and Karpinski. The Hindu-Arabic Numerals. Ginn, $1.40.
8. White. Scrap-book of Elementary Mathematics. Open Court Pub. Co., $1.00.
9. School Science and Mathematics.
   Smith and Turton. (Magazine and membership in Central Association of Science and Mathematics Teachers, $2.50 per year. Chicago.)
10. *Teachers' College Record* — Columbia University, $1.50 per year.
11. *American Mathematical Monthly*, Journal of Mathematical Association of America. (Magazine and membership, $3.00 per year. Chicago.)
12. Smith. Number Stories of Long Ago. Ginn & Co. $0.48.

B. TABLES

Linear Measure for Length

12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
$5\frac{1}{2}$ yards or $16\frac{1}{2}$ feet = 1 rod (rd.)
320 rods = 1 mile (mi.)
1 mi. = 320 rd. = 1760 yd. = 5280 ft. = 63,360 in.
A hand = 4 in., used in measuring the height of horses.
A fathom = 6 ft., used in measuring the depth of large bodies of water.
A knot = 1.15 mi., used in measuring distances at sea.

Square Measure for Surface

144 square inches (sq. in.) = 1 square foot (sq. ft.)
9 square feet = 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards = 1 square rod (sq. rd.)
160 square rods = 1 acre (A.)
640 acres = 1 square mile (sq. mi.)
1 A. = 160 sq. rd. = 4840 sq. yd. = 43,560 sq. ft.
A section = 1 square mile
A square = 100 square feet, used in roofing, flooring, and painting.

Cubic Measure for Volume

1728 cubic inches (cu. in.) = 1 cubic foot (cu. ft.)
27 cubic feet = 1 cubic yard (cu. yd.)

A cord = 128 cu. ft., used in measuring wood. It is usually a pile 8 ft. by 4 ft. by 4 ft.

Measure of Capacity

A. *Liquid Measure*

4 gills (gi.) = 1 pint (pt.)
2 pints = 1 quart (qt.)
4 quarts = 1 gallon (gal.)

1 gal. = 231 cu. in.
1 barrel (bbl.) = $31\frac{1}{2}$ gal.
1 hogshead = 63 gal.

In commerce barrels and hogsheads vary in size.
1 gallon of water weighs about $8\frac{1}{2}$ pounds.
1 cubic foot of water weighs about $62\frac{1}{2}$ pounds.

B. *Dry Measure for Fruits, Vegetables, and Grain*

2 pints = 1 quart (qt.)
8 quarts = 1 peck (pk.)
4 pecks = 1 bushel (bu.)

1 bushel = 2150.42 cu. in. or about $1\frac{1}{4}$ cu. ft.

Avoirdupois Weight

16 ounces (oz.) = 1 pound (lb.)
2000 pounds = 1 ton (T.)
1 hundredweight (cwt.) = 100 pounds.
1 pound avoirdupois = 7000 grains.

1 long ton = 2240 pounds, used in weighing coal and ores at the mines.

Standard Weights

1 bu. of wheat	= 60 lb.	1 bu. of white potatoes	= 60 lb.
1 bu. of corn, in the ear	= 70 lb.	1 bu. of sweet potatoes	= 55 lb.
1 bu. of corn, shelled	= 56 lb.	1 bu. of corn meal.	= 48 lb.
1 bu. of oats......	= 32 lb.	1 bu. of clover seed	= 60 lb.
1 bu. of rye.......	= 56 lb.	1 bbl. of flour.....	= 196 lb.
1 bu. of barley....	= 48 lb.	1 bbl. of pork.....	= 200 lb.

1 keg of nails...... = 100 lb.

Troy Weights for Jewels and Precious Metals

24 grains (gr.) = 1 pennyweight (pwt.)
20 pennyweight = 1 ounce (oz.)
12 ounces = 1 pound (lb.)
1 pound troy = 5760 grains

Time

60 seconds (sec.) = 1 minute (min.)
60 minutes = 1 hour (hr.)
24 hours = 1 day (da.)
7 days = 1 week (wk.)
365 days = 1 year (yr.)

A leap year = 366 days.

A business year usually is 360 days or 12 months of 30 days each.

A decade = 10 years.

A century = 100 years.

Counting

12 things = 1 dozen (doz.)
12 dozen = 1 gross
A score = 200 things

Paper

24 sheets = 1 quire
20 quires = 1 ream

Paper is usually sold by the 1000 (M), by 500 (D), or by the 100 (C) sheets.

In practice 1 ream = 500 sheets
1 quire = 25 envelopes or cards

Arcs of a Circle

60 seconds (″) = 1 minute (′)
60 minutes = 1 degree (°)
360 degrees = 1 circumference.

Angles

60 seconds (″) = 1 minute (′)
60 minutes = 1 degree (°)
90 degrees = 1 right angle (rt. ∠) or 1 quadrant
180 degrees = 1 straight angle (st. ∠)
360 degrees = 1 perigon

United States Money

10 mills = 1 cent (¢)
10 cents = 1 dime
10 dimes = 1 dollar ($)

Money Equivalents

England — 1 pound (£) = $4.8665 in U. S. gold coin
France — 1 franc (fr.) = 19.3 cents

Metric System

Linear Measure for Length

10 millimeters (mm.) = 1 centimeter (cm.)
10 centimeters = 1 decimeter (dm.)
10 decimeters = 1 meter (m.)
1 meter = 39.37 inches
1 yard = .9144 meter

Square Measure for Surface

100 square millimeters (sq. mm.) = 1 square centimeter (sq. cm.)
100 square centimeters = 1 square decimeter (sq. dm.)
100 square decimeters = 1 square meter (sq. m.)
1 square meter = 1.196 square yards
1 square yard = .836 square meter

Cubic Measure for Volume

1000 cubic millimeters (cu. mm.)	= 1 cubic centimeter (cu. cm. or c.c.)
1000 cubic centimeters	= 1 cubic decimeter (cu. dm.)
1000 cubic decimeters	= 1 cubic meter (cu. m.)
1 cubic meter	= 1.308 cubic yards
1 cubic yard	= .765 cubic meter

Measure of Capacity

1 liter (l.)	= .908 dry quart
1 dry quart	= 1.1012 liters
1 liter	= 1.0567 liquid quarts
1 liquid quart	= .94636 liter

Measure of Weight

1 gram (g.)	= weight of 1 cu. cm. of water
	= .0022 pound
1 pound	= 453.59 grams

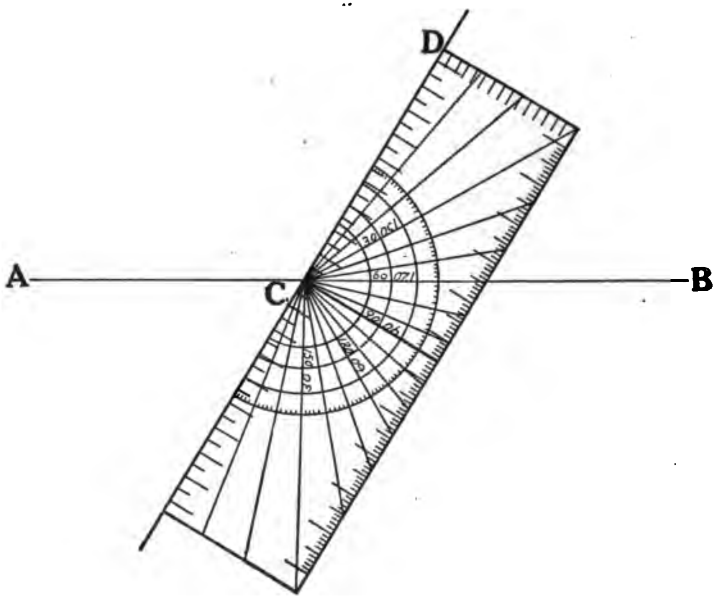
C. A PROTRACTOR AND ITS USE

There are many kinds of protractors but the most satisfactory kind is a combined ruler and protractor. It should be made of celluloid because of its transparency. One edge of the ruler should be marked in English units and the other in metric units. One end should be marked in tenths of inches.

The Granville Combined Ruler and Protractor has one end marked in fiftieths of an inch and its width is exactly the square root of 2 inches.

The following figure shows how to draw an angle of 60°.

Given any line AB. Select some point, C, in the line and place the center of the protractor at this point. Slide the protractor around until its 60° line exactly coincides with AB. Then draw a line along the edge CD. $\angle BCD$ is a 60° angle.



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